10.3 Exercises

1–4 □ Set up, but do not evaluate, an integral that represents the length of the curve.

1. $x = t - t^2,\quad y = \frac{4}{3}t^{3/2},\quad 1 \leqslant t \leqslant 2$

2. $x = 1 + e^t,\quad y = t^2,\quad -3 \leqslant t \leqslant 3$

3. $x = t\sin t,\quad y = t\cos t,\quad 0 \leqslant t \leqslant \pi/2$

4. $x = \ln t,\quad y = \sqrt{t+1},\quad 1 \leqslant t \leqslant 5$

5–8 □ Find the length of the curve.

5. $x = t^3,\quad y = t^2,\quad 0 \leqslant t \leqslant 4$

6. $x = a(\cos\theta + \theta\sin\theta),\quad y = a(\sin\theta - \theta\cos\theta),$ $0 \leqslant \theta \leqslant \pi$

7. $x = \dfrac{t}{1+t},\quad y = \ln(1+t),\quad 0 \leqslant t \leqslant 2$

8. $x = e^t + e^{-t},\quad y = 5 - 2t,\quad 0 \leqslant t \leqslant 3$

9–11 □ Graph the curve and find its length.

9. $x = e^t\cos t,\quad y = e^t\sin t,\quad 0 \leqslant t \leqslant \pi$

10. $x = 3t - t^3,\quad y = 3t^2,\quad 0 \leqslant t \leqslant 2$

11. $x = e^t - t,\quad y = 4e^{t/2},\quad -8 \leqslant t \leqslant 3$

12. Graph the curve

$$x = t\cos t + \sin t \qquad y = t\sin t - \cos t \qquad -\pi \leqslant t \leqslant \pi$$

Then use a CAS or a table of integrals to find the exact length of the curve.

13. Use Simpson's Rule with $n = 10$ to estimate the length of the curve $x = \ln t$, $y = e^{-t}$, $1 \leqslant t \leqslant 2$.

14. In Exercise 37 in Section 10.1 you were asked to derive the parametric equations $x = 2a\cot\theta$, $y = 2a\sin^2\theta$ for the curve called the witch of Maria Agnesi. Use Simpson's Rule with $n = 4$ to estimate the length of the arc of this curve given by $\pi/4 \leqslant \theta \leqslant \pi/2$.

15–16 □ Find the distance traveled by a particle with position (x, y) as t varies in the given time interval. Compare with the length of the curve.

15. $x = \sin^2 t,\quad y = \cos^2 t,\quad 0 \leqslant t \leqslant 3\pi$

16. $x = \cos^2 t,\quad y = \cos t,\quad 0 \leqslant t \leqslant 4\pi$

17. Show that the total length of the ellipse $x = a\sin\theta$, $y = b\cos\theta$, $a > b > 0$, is

$$L = 4a\int_0^{\pi/2}\sqrt{1 - e^2\sin^2\theta}\,d\theta$$

where e is the eccentricity of the ellipse ($e = c/a$, where $c = \sqrt{a^2 - b^2}$).

18. Find the total length of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$.

19. (a) Graph the epitrochoid with equations

$$x = 11\cos t - 4\cos(11t/2)$$
$$y = 11\sin t - 4\sin(11t/2)$$

What parameter interval gives the complete curve?

(b) Use your CAS to find the approximate length of this curve.

20. A curve called **Cornu's spiral** is defined by the parametric equations

$$x = C(t) = \int_0^t \cos(\pi u^2/2)\,du$$
$$y = S(t) = \int_0^t \sin(\pi u^2/2)\,du$$

where C and S are the Fresnel functions that were introduced in Chapter 5.

(a) Graph this curve. What happens as $t \to \infty$ and as $t \to -\infty$?

(b) Find the length of Cornu's spiral from the origin to the point with parameter value t.

21–22 □ Set up, but do not evaluate, an integral that represents the area of the surface obtained by rotating the given curve about the x-axis.

21. $x = t^3,\quad y = t^4,\quad 0 \leqslant t \leqslant 1$

22. $x = \sin^2 t,\quad y = \sin 3t,\quad 0 \leqslant t \leqslant \pi/3$

23–25 □ Find the area of the surface obtained by rotating the given curve about the x-axis.

23. $x = t^3,\quad y = t^2,\quad 0 \leqslant t \leqslant 1$

24. $x = 3t - t^3,\quad y = 3t^2,\quad 0 \leqslant t \leqslant 1$

25. $x = a\cos^3\theta,\quad y = a\sin^3\theta,\quad 0 \leqslant \theta \leqslant \pi/2$

26. Graph the curve

$$x = 2\cos\theta - \cos 2\theta \qquad y = 2\sin\theta - \sin 2\theta$$

If this curve is rotated about the x-axis, find the area of the resulting surface. (Use your graph to help find the correct parameter interval.)

27. If the curve

$$x = t + t^3 \qquad y = t - \frac{1}{t^2} \qquad 1 \leqslant t \leqslant 2$$

is rotated about the x-axis, estimate the area of the resulting surface to three decimal places. (If your calculator or CAS evaluates definite integrals numerically, use it. Otherwise, use Simpson's Rule.)

28. If the arc of the curve in Exercise 14 is rotated about the x-axis, estimate the area of the resulting surface using Simpson's Rule with $n = 4$.

29–30 □ Find the surface area generated by rotating the given curve about the y-axis.

29. $x = 3t^2$, $y = 2t^3$, $0 \leqslant t \leqslant 5$

30. $x = e^t - t$, $y = 4e^{t/2}$, $0 \leqslant t \leqslant 1$

31. Find the surface area of the ellipsoid obtained by rotating the ellipse $x = a\cos\theta$, $y = b\sin\theta$ $(a > b)$ about (a) the x-axis and (b) the y-axis.

32. Use Formula 10.2.2 to derive Formula 5 from Formula 8.2.5 for the case in which the curve can be represented in the form $y = F(x)$, $a \leqslant x \leqslant b$.

33. The **curvature** at a point P of a curve is defined as

$$\kappa = \left|\frac{d\phi}{ds}\right|$$

where ϕ is the angle of inclination of the tangent line at P, as shown in the figure. Thus, the curvature is the absolute value of the rate of change of ϕ with respect to arc length. It can be regarded as a measure of the rate of change of direction of the curve at P and will be studied in greater detail in Chapter 13.

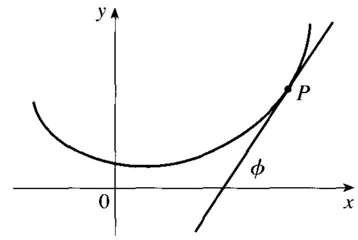
(a) For a parametric curve $x = x(t)$, $y = y(t)$, derive the formula

$$\kappa = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

where the dots indicate derivatives with respect to t, so $\dot{x} = dx/dt$. [*Hint:* Use $\phi = \tan^{-1}(dy/dx)$ and Equation 10.2.2 to find $d\phi/dt$. Then use the Chain Rule to find $d\phi/ds$.]

(b) By regarding a curve $y = f(x)$ as the parametric curve $x = x$, $y = f(x)$, with parameter x, show that the formula in part (a) becomes

$$\kappa = \frac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}}$$



34. (a) Use the formula in Exercise 33(b) to find the curvature of the parabola $y = x^2$ at the point $(1, 1)$.
(b) At what point does this parabola have maximum curvature?

35. Use the formula in Exercise 33(a) to find the curvature of the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ at the top of one of its arches.

36. (a) Show that the curvature at each point of a straight line is $\kappa = 0$.
(b) Show that the curvature at each point of a circle of radius r is $\kappa = 1/r$.

10.4 Polar Coordinates

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. So far we have been using Cartesian coordinates, which are directed distances from two perpendicular axes. In this section we describe a coordinate system introduced by Newton, called the **polar coordinate system**, which is more convenient for many purposes.

We choose a point in the plane that is called the **pole** (or origin) and is labeled O. Then we draw a ray (half-line) starting at O called the **polar axis**. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.

If P is any other point in the plane, let r be the distance from O to P and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in Figure 1. Then the point P is represented by the ordered pair (r, θ) and r, θ are called **polar coordi-**

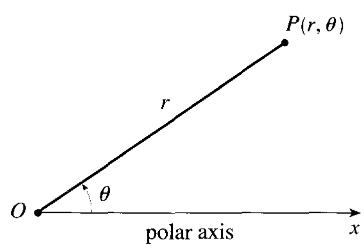


FIGURE 1