On the other hand, the average value of the velocity function on the interval is

post-answers

$$v_{ave} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} v(t)\,dt = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} s'(t)\,dt$$

$$= \frac{1}{t_2 - t_1}[s(t_2) - s(t_1)] \qquad \text{(by the Total Change Theorem)}$$

$$= \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \text{average velocity}$$

□

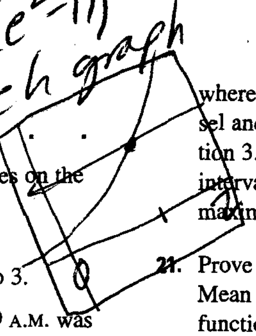
6.5 Exercises

1–8 □ Find the average value of the function on the given interval.

1. $f(x) = x^2$, $[-1, 1]$
2. $f(x) = 1/x$, $[1, 4]$
3. $g(x) = \cos x$, $[0, \pi/2]$
4. $g(x) = \sqrt{x}$, $[1, 4]$
5. $f(t) = te^{-t^2}$, $[0, 5]$
6. $f(\theta) = \sec\theta\tan\theta$, $[0, \pi/4]$
7. $h(x) = \cos^4 x \sin x$, $[0, \pi]$
8. $h(r) = 3/(1 + r)^2$, $[1, 6]$ 3/14

9–12 □
(a) Find the average value of f on the given interval.
(b) Find c such that $f_{ave} = f(c)$.
(c) Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

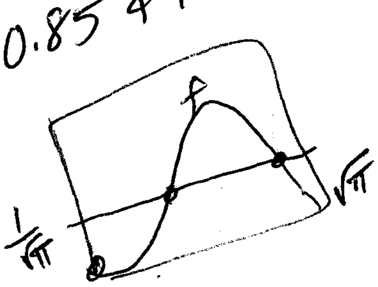
9. $f(x) = 4 - x^2$, $[0, 2]$
10. ~~$f(x) = e^x$, $[0, 2]$~~ (a) $\frac{1}{2}(e^2-1)$ (b) $\ln\left(\frac{1}{2}(e^2-1)\right)$ (c). sketch graph
11. $f(x) = x^3 - x + 1$, $[0, 2]$
12. $f(x) = x\sin(x^2)$, $[0, \sqrt{\pi}]$



13. If f is continuous and $\int_1^3 f(x)\,dx = 8$, show that f takes on the value 4 at least once on the interval $[1, 3]$.
14. Find the numbers b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.
15. In a certain city the temperature (in °F) t hours after 9 A.M. was approximated by the function
$$T(t) = 50 + 14\sin\frac{\pi t}{12}$$
Find the average temperature during the period from 9 A.M. to 9 P.M.

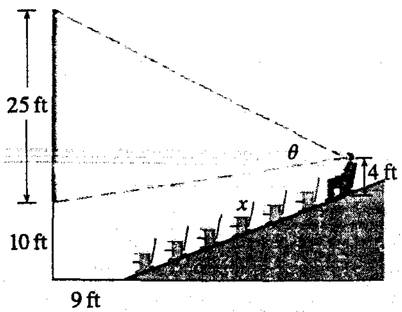
(a) $\frac{1}{\sqrt{\pi}}$ (b) $x \approx 0.85$ & 1.67

(c)



16. The temperature of a metal rod, 5 m long, is $4x$ (in °C) at a distance x meters from one end of the rod. What is the average temperature of the rod?
17. The linear density in a rod 8 m long is $12/\sqrt{x+1}$ kg/m, where x is measured in meters from one end of the rod. Find the average density of the rod.
18. If a freely falling body starts from rest, then its displacement is given by $s = \frac{1}{2}gt^2$. Let the velocity after a time T be v_T. Show that if we compute the average of the velocities with respect to t we get $v_{ave} = \frac{1}{2}v_T$, but if we compute the average of the velocities with respect to s we get $v_{ave} = \frac{2}{3}v_T$.
19. Use the result of Exercise 75 in Section 5.5 to compute the average volume of inhaled air in the lungs in one respiratory cycle.
20. The velocity v of blood that flows in a blood vessel with radius R and length l at a distance r from the central axis is
$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$
where P is the pressure difference between the ends of the vessel and η is the viscosity of the blood (see Example 7 in Section 3.3). Find the average velocity (with respect to r) over the interval $0 \leq r \leq R$. Compare the average velocity with the maximum velocity.
21. Prove the Mean Value Theorem for Integrals by applying the Mean Value Theorem for derivatives (see Section 4.2) to the function $F(x) = \int_a^x f(t)\,dt$.
22. If $f_{ave}[a, b]$ denotes the average value of f on the interval $[a, b]$ and $a < c < b$, show that
$$f_{ave}[a, b] = \frac{c - a}{b - a} f_{ave}[a, c] + \frac{b - c}{b - a} f_{ave}[c, b]$$

Applied Project — Where to Sit at the Movies



A movie theater has a screen that is positioned 10 ft off the floor and is 25 ft high. The first row of seats is placed 9 ft from the screen and the rows are set 3 ft apart. The floor of the seating area is inclined at an angle of $\alpha = 20°$ above the horizontal and the distance up the incline that you sit is x. The theater has 21 rows of seats, so $0 \leqslant x \leqslant 60$. Suppose you decide that the best place to sit is in the row where the angle θ subtended by the screen at your eyes is a maximum. Let's also suppose that your eyes are 4 ft above the floor, as shown in the figure. (In Exercise 54 in Section 4.7 we looked at a simpler version of this problem, where the floor is horizontal, but this project involves a more complicated situation and requires technology.)

1. Show that

$$\theta = \arccos\left(\frac{a^2 + b^2 - 625}{2ab}\right)$$

where

$$a^2 = (9 + x\cos\alpha)^2 + (31 - x\sin\alpha)^2$$

and

$$b^2 = (9 + x\cos\alpha)^2 + (x\sin\alpha - 6)^2$$

2. Use a graph of θ as a function of x to estimate the value of x that maximizes θ. In which row should you sit? What is the viewing angle θ in this row?

3. Use your computer algebra system to differentiate θ and find a numerical value for the root of the equation $d\theta/dx = 0$. Does this value confirm your result in Problem 2?

4. Use the graph of θ to estimate the average value of θ on the interval $0 \leqslant x \leqslant 60$. Then use your CAS to compute the average value. Compare with the maximum and minimum values of θ.

6 Review

CONCEPT CHECK

1. (a) Draw two typical curves $y = f(x)$ and $y = g(x)$, where $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$. Show how to approximate the area between these curves by a Riemann sum and sketch the corresponding approximating rectangles. Then write an expression for the exact area.
   (b) Explain how the situation changes if the curves have equations $x = f(y)$ and $x = g(y)$, where $f(y) \geqslant g(y)$ for $c \leqslant y \leqslant d$.

2. Suppose that Sue runs faster than Kathy throughout a 1500-meter race. What is the physical meaning of the area between their velocity curves for the first minute of the race?

3. (a) Suppose S is a solid with known cross-sectional areas. Explain how to approximate the volume of S by a Riemann sum. Then write an expression for the exact volume.
   (b) If S is a solid of revolution, how do you find the cross-sectional areas?

4. (a) What is the volume of a cylindrical shell?
   (b) Explain how to use cylindrical shells to find the volume of a solid of revolution.
   (c) Why might you want to use the shell method instead of slicing?

5. Suppose that you push a book across a 6-meter-long table by exerting a force $f(x)$ at each point from $x = 0$ to $x = 6$. What does $\int_0^6 f(x)\,dx$ represent? If $f(x)$ is measured in newtons, what are the units for the integral?

6. (a) What is the average value of a function f on an interval $[a, b]$?
   (b) What does the Mean Value Theorem for Integrals say? What is its geometric interpretation?

EXERCISES

1–6 □ Find the area of the region bounded by the given curves.

1. $y = x^2 - x - 6, \quad y = 0$ $\frac{125}{6}$
2. $y = 20 - x^2, \quad y = x^2 - 12$
3. $y = e^x - 1, \quad y = x^2 - x, \quad x = 1$ $e - 11/6$
4. $x - 2y + 7 = 0, \quad y^2 - 6y - x = 0$
5. $y = \sin x, \quad y = -\cos x, \quad x = 0, \quad x = \pi$

$2\sqrt{2}$

6. $y = x^3$, $y = x^2 - 4x + 4$, $x = 0$, $x = 2$

7–11 □ Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.

7. $y = \sqrt{x - 1}$, $y = 0$, $x = 3$; about the x-axis 2π

8. $y = e^{-2x}$, $y = 1 + x$, $x = 1$; about the x-axis

9. $x + 3 = 4y - y^2$, $x = 0$; about the x-axis $\frac{16\pi}{3}$

10. $y = x^3$, $y = 8$, $x = 0$; about the y-axis

11. $x^2 - y^2 = a^2$, $x = a + h$ (where $a > 0, h > 0$); about the y-axis $\frac{4}{3}\pi(2ah+h^2)^{3/2}$

12–14 □ Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.

12. $y = \cos x$, $y = 0$, $x = 3\pi/2$, $x = 5\pi/2$; about the y-axis

13. $y = x^3$, $y = x^2$; about $y = 1$ $\int \pi[(1-x^3)^2-(1-x^2)^2]\,dx$

14. $y = x^3$, $y = 8$, $x = 0$; about $x = 2$

15. Find the volumes of the solids obtained by rotating the region bounded by the curves $y = x$ and $y = x^2$ about the following lines:
(a) The x-axis $2\pi/15$ (b) The y-axis $\pi/6$ (c) $y = 2$ $8\pi/15$

16. Let $\mathfrak{R}$ be the region in the first quadrant bounded by the curves $y = x^3$ and $y = 2x - x^2$. Calculate the following quantities:
(a) The area of $\mathfrak{R}$
(b) The volume obtained by rotating $\mathfrak{R}$ about the x-axis
(c) The volume obtained by rotating $\mathfrak{R}$ about the y-axis

17. Let $\mathfrak{R}$ be the region bounded by the curves $y = \tan(x^2)$, $x = 1$, and $y = 0$. Use the Midpoint Rule with $n = 4$ to estimate the following.
(a) The area of $\mathfrak{R}$
(b) The volume obtained by rotating $\mathfrak{R}$ about the x-axis

18. Let $\mathfrak{R}$ be the region bounded by the curves $y = 1 - x^2$ and $y = x^6 - x + 1$. Estimate the following quantities:
(a) The x-coordinates of the points of intersection of the curves
(b) The area of $\mathfrak{R}$
(c) The volume generated when $\mathfrak{R}$ is rotated about the x-axis
(d) The volume generated when $\mathfrak{R}$ is rotated about the y-axis

19–22 □ Each integral represents the volume of a solid. Describe the solid.

19. $\int_0^{\pi/2} 2\pi x \cos x \, dx$

20. $\int_0^{\pi/2} 2\pi \cos^2 x \, dx$

21. $\int_0^{2} 2\pi y(4 - y^2) \, dy$

22. $\int_0^{1} \pi[(2 - x^2)^2 - (2 - \sqrt{x})^2] \, dx$

23. The base of a solid is a circular disk with radius 3. Find the volume of the solid if ~~parallel~~ cross-sections perpendicular to the base are isosceles right triangles with hypotenuse lying along the base. 36

24. The base of a solid is the region bounded by the parabolas $y = x^2$ and $y = 2 - x^2$. Find the volume of the solid if the cross-sections perpendicular to the x-axis are squares with one side lying along the base. $64/15$

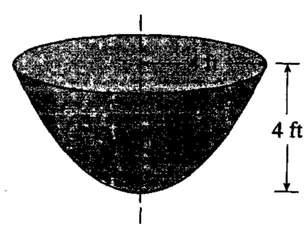
25. The height of a monument is 20 m. A ~~horizontal~~ cross-section at a distance x meters from the top is an equilateral triangle with side $x/4$ meters. Find the volume of the monument. $\frac{125\sqrt{3}}{3}\,m^3$

26. (a) The base of a solid is a square with vertices located at $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$. Each cross-section perpendicular to the x-axis is a semicircle. Find the volume of the solid. $\pi/3$
(b) Show that by cutting the solid of part (a), we can rearrange it to form a cone. Thus compute its volume more simply. $\pi/3$

27. A force of 30 N is required to maintain a spring stretched from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 20 cm?

28. A 1600-lb elevator is suspended by a 200-ft cable that weighs 10 lb/ft. How much work is required to raise the elevator from the basement to the third floor, a distance of 30 ft?

29. A tank full of water has the shape of a paraboloid of revolution as shown in the figure; that is, its shape is obtained by rotating a parabola about a vertical axis.
(a) If its height is 4 ft and the radius at the top is 4 ft, find the work required to pump the water out of the tank.
(b) After 4000 ft-lb of work has been done, what is the depth of the water remaining in the tank?



30. Find the average value of the function $f(x) = x^2\sqrt{1 + x^3}$ on the interval $[0, 2]$. $26/9$

31. If f is a continuous function, what is the limit as $h \to 0$ of the average value of f on the interval $[x, x + h]$?

32. Let $\mathfrak{R}_1$ be the region bounded by $y = x^2$, $y = 0$, and $x = b$, where $b > 0$. Let $\mathfrak{R}_2$ be the region bounded by $y = x^2$, $x = 0$, and $y = b^2$.
(a) Is there a value of b such that $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have the same area?
(b) Is there a value of b such that $\mathfrak{R}_1$ sweeps out the same volume when rotated about the x-axis and the y-axis?
(c) Is there a value of b such that $\mathfrak{R}_1$ and $\mathfrak{R}_2$ sweep out the same volume when rotated about the x-axis?
(d) Is there a value of b such that $\mathfrak{R}_1$ and $\mathfrak{R}_2$ sweep out the same volume when rotated about the y-axis?