Taylor Polynomials and Infinite Series
A chapter for a first-year calculus course

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Preface

The subject of infinite series is delightful in its richness and beauty. From the "basic" concept that adding together infinitely many things can sometimes lead to a finite result, to the extremely powerful idea that functions can be represented by Taylor series–"infinomials," as one of my students once termed them–this corner of the calculus curriculum covers an assortment of topics that I have always found fascinating. This text is a complete, stand-alone chapter covering infinite sequences and series, Taylor polynomials, and power series that attempts to make these wonderful topics accessible and understandable for high school students in an introductory calculus course. It can be used to replace the analogous chapter or chapters in a standard calculus textbook, or it can be used as a secondary resource for students who would like an alternate narrative explaining how these topics are connected.

There are countless calculus textbooks on the market, and it is the good fortune of calculus students and teachers that many of them are quite good. You almost have to go out of your way to pick a bad introductory calculus text. Why then am I adding to an already-glutted market my own chapter on Taylor polynomials and Taylor series? For my tastes, none of the available excellent texts organize or present these topics quite the way I would like. Not without reason, many calculus students (and not a small number of their teachers) find infinite series challenging. They deserve a text that highlights the big ideas and focuses their attention on the key themes and concepts.

The organization of this chapter differs from that of standard texts. Rather than progress linearly from a starting point of infinite sequences, pass through infinite series and convergence tests, and conclude with power series generally and Taylor series specifically, my approach is almost the opposite. The reason for the standard approach is both simple and clear; it is the only presentation that makes sense if the goal is to develop the ideas with mathematical rigor. The danger, however, is that by the time the students have slogged through a dozen different convergence tests to arrive at Taylor series they may have lost sight of what is important: that functions can be approximated by polynomials and represented exactly by "infinitely long polynomials."

So I start with the important stuff first, at least as much of it as I can get away with. After an obligatory and light-weight section reviewing sequences and series topics from precalculus, I proceed directly to Taylor polynomials. Taylor polynomials are an extension of linearization functions, and they are a concrete way to frame the topics that are to come in the rest of the chapter. The first half of the chapter explores Taylor polynomials—how to build them, under what conditions they provide good estimates, error approximation—always with an eye to the Taylor series that are coming down the road. The second half of the chapter extends naturally from polynomials to power series, with convergence tests entering the scene on an as-needed basis. The hope is that this organizational structure will keep students thinking about the big picture. Along the way, the book is short on proofs and long on ideas. There is no attempt to assemble a rigorous development of the theory; there are plenty of excellent textbooks on the market that already do that. Furthermore, if we are to be honest with each other, students who want to study these topics with complete mathematical precision will need to take an additional course in advanced calculus anyway.

A good text is a resource, both for the students and the teacher. For the students, I have given this chapter a conversational tone which I hope makes it readable. Those of us who have been teaching a while know that students don't always read their textbooks as much as we'd like. (They certainly don't read prefaces, at any rate. If you are a student reading this preface, then you are awesome.) Even so, the majority of the effort of this chapter was directed toward crafting an exposition that makes sense of the big ideas and how they fit together. I hope that your students will read it and find it helpful. For teachers, a good text provides a bank of problems. In addition to standard problem types, I have tried to include problems that get at the bigger concepts, that challenge students to think in accordance with the rule of four (working with mathematical concepts verbally, numerically, graphically, and analytically), and that provide good preparation for that big exam that occurs in early May. The final role a math text should fill if it is to be a good resource is to maintain a sense of the development of the subject, presenting essential theorems in a logical order and supporting them with rigorous proofs. This chapter doesn't do that. Fortunately, students have their primary text, the one they have been using for the rest of their calculus course. While the sequencing of the topics in this chapter is very different from that of a typical text, hopefully the student interested in hunting down a formal proof for a particular theorem or convergence test will be able to find one in his or her other book.

Using this chapter ought to be fairly straightforward. As I have said, the focus of the writing was on making it user-friendly for students. The overall tone is less formal, but definitions and theorems are given precise, correct wording and are boxed for emphasis. There are numerous example problems throughout the sections. There are also practice problems. Practices are just like examples, except that their solutions are delayed until the end of the section. Practices often follow examples, and the hope is that students will work the practice problems as they read to ensure that they are picking up the essential ideas and skills. The end of the solution to an example problem (or to the proof of a theorem) is signaled by a ◊. Parts of the text have been labeled as "optional." The optional material is not covered on the AP test, nor is it addressed in the problem sets at the ends of the sections. This material can be omitted without sacrificing the coherence of the story being told by the rest of the chapter.

There are several people whose contributions to this chapter I would like to acknowledge. First to deserve thanks is a man who almost certainly does not remember me as well as I remember him. Robert Barefoot was the first person (though not the last!) to suggest to me that this part of the calculus curriculum should be taught with Taylor polynomials appearing first, leaving the library of convergence tests for later. Much of the organization of this chapter is influenced by the ideas he presented at a workshop I attended in 2006. As for the actual creation of this text, many people helped me by reading drafts, making comments, and discussing the presentation of topics in this chapter. I am very grateful to Doug Kühlmann, Melinda Certain, Phil Certain, Beth Gallis, Elisse Ghitelman, Carl LaCombe, Barbara De Roes, James King, and Scott Barcus for their thoughtful comments, feedback, error corrections, and general discussion on the composition and use of the chapter. Scott was also the first to field test the chapter, sharing it with his students before I even had a chance to share it with mine. I am certainly in debt to my wife Laura who proof-read every word of the exposition and even worked several of the problems. Finally, I would be remiss if I did not take the time to thank Mary Lappan, Steve Viktora, and Garth Warner. Mary and Steve were my first calculus teachers, introducing me to a subject that I continue to find more beautiful and amazing with every year that I teach it myself. Prof. Warner was the real analysis professor who showed me how calculus can be made rigorous; much of my current understanding of series is thanks to him and his course.

The subject of infinite series can be counter-intuitive and even bizarre, but it is precisely this strangeness that has continued to captivate me since I first encountered it. Taylor series in particular I find to be nothing less than the most beautiful topic in the high school mathematics curriculum. Whether you are a student or a teacher, I hope that this chapter enables you to enjoy series as much as I do.

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Section 1 – Review of Sequences and Series

This chapter is principally about two things: Taylor polynomials and Taylor series. Taylor polynomials are a logical extension of linearization (a.k.a. tangent line approximations), and they will provide you with a good opportunity to extend what you have already learned about calculus. Taylor series, in turn, extend the idea of Taylor polynomials. Taylor series have additional appeal in the way they tie together many different topics in mathematics in a surprising and, in my opinion, amazing way.

Before we can dive in to the beauty of Taylor polynomials and Taylor series, we need to review some fundamentals about sequences and series, topics you should have studied in your precalculus course. Most of this will (hopefully) look familiar to you, but a quick refresher is not a bad thing. One disclaimer: sequences and series are rich topics with many nuances and opportunities for exploration and further study. We won't do either subject justice in this section. Our goal now is just to make sure we have all the tools we need to hit the ground running.

Sequences

A sequence, simply put, is just a list of numbers where the numbers are counted by some index variable. We often use *i*, *j*, *k*, or *n* for the index variable. Here are a couple simple examples of sequences:

$$\{a_n\} = \{1, 2, 3, 4, 5, \ldots\}$$
$$\{b_k\} = \{3, 8, -2, 5, 7, \ldots\}$$

For the sequence $\{a_n\}$, if we assume that the initial value of the index variable (n) is 1, then a_2 is 2, a_5 is 5, and so on. We could guess that the pattern in a_n will continue and that $a_n = n$ for any whole number n. So a_{24} is probably 24. However, we often want to make the index variable start at 0 instead of 1. If we do that for $\{a_n\}$, then we have $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, $a_{24} = 25$, and so on. With this initial value of the index, an expression for the **general term** of the sequence would have to be $a_n = n + 1$. Most of the time we will want to start the index at 0, unless there is some reason why we can't. But if no initial value of the index is specified, then it is up to you to choose an initial value that makes sense to you. All of this is moot with the sequence $\{b_k\}$; there is no obvious pattern to the terms in this sequence (or at least none was intended), so we cannot come up with an expression to describe the general term.

(Side note: The difference between the symbols $\{a_n\}$ and a_n is similar to the difference between the symbols *f* and $f(x)$ for regular functions. While *f* denotes the set of all ordered pairs making up the function, $f(x)$ is the output value of *f* at some number *x*. Similarly, $\{a_n\}$ denotes the sequence as a whole; a_n is the value of the sequence at *n*.)

A more technical definition of the word sequence is provided below.

Definition: A **sequence** is a function whose domain is either the positive or the non-negative integers.

Go back and look at the sequences $\{a_n\}$ and $\{b_k\}$ (especially $\{a_n\}$). Do you see how they are in fact functions mapping the whole numbers to other numbers? We could have written $a(n)$ and $b(k)$, but for whatever reason we don't.

Example 1

Write the first 5 terms of the sequence $a_n = \dfrac{(-1)^n}{n^2}$.

Solution
We cannot use 0 for the initial value of n because then we would be dividing by 0. So let the first value for n be 1. Then the first five terms of the sequence are $\frac{(-1)^1}{1^2}, \frac{(-1)^2}{2^2}, \frac{(-1)^3}{3^2}, \frac{(-1)^4}{4^2}, \frac{(-1)^5}{5^2}$. This simplifies to $-1, \frac{1}{4}, \frac{-1}{9}, \frac{1}{16}, \frac{-1}{25}$. Notice how the factor $(-1)^n$ caused the terms of the sequence to alternate in sign. This will come up again and again and again for the rest of this chapter. So here is a question for you. How would we have needed to change the $(-1)^n$ if we wanted the terms to be positive, negative, positive, … instead of negative, positive, negative, …? ◊

Example 2

Give an expression for the general term of the sequence $\{a_n\} = \left\{1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \ldots\right\}$.

Solution
We started with an initial index value of 1 in Example 1, so for variety let's use 0 this time. When the terms of a sequence are fractions, as they are here, it makes sense to model the numerators and denominators separately. The numerators are just the whole numbers 1, 2, 3,…. It would be simplest to just say that the numerator of the n^{th} term is n, but since we chose to start n at 0, we have to account for that. The numerators are 0+1, 1+1, 2+1, etc.; an expression for the numerators is $n+1$. (It seems silly to have started n at 0 instead of 1, but wait for the denominator.) There are many ways to think about the denominators. For one, the denominators are increasing by 2 from one term to the next, so they can be modeled with a linear function whose slope is 2. The initial denominator is 1 (in 1/1), so we can simply write $2n+1$ for the denominator. Another approach is to recognize that the denominators are odd numbers. Odd numbers are numbers that leave a remainder of 1 when divided by 2; in other words, they are numbers that are one more than (or one less than) even numbers. Even numbers can be expressed by $2n$, where n is an integer, so the odds can be expressed as $2n+1$. Coming up with expressions for even numbers and for odd numbers will be another recurring theme in the chapter. In any event, the general term can be expressed as $a_n = \frac{n+1}{2n+1}$. ◊

Practice 1

Starting with $n = 0$, write the first five terms of the sequence $a_n = \frac{1}{n!}$. (Recall that, by definition, $0! = 1$.)*

Practice 2

Write an expression for the general term of the sequence $\left\{\frac{1}{2}, \frac{8}{3}, \frac{27}{4}, \frac{64}{5}, \ldots\right\}$.

Practice 3
Revisit Example 2. Rewrite the general term of the sequence if the initial value of n is 1 instead of 0.

If $\lim_{n\to\infty} a_n$ equals some number L, then we say that the sequence $\{a_n\}$ **converges** to L. Strictly speaking, this is a bad definition as it stands; we have not defined limits for sequences, and you have probably only seen limits defined for *continuous* functions. Sequences, by contrast, are definitely not continuous. But the idea is exactly the same. We want to know what happens to a_n as n gets big. If the terms become arbitrarily close to a particular real number, then the sequence converges to that number.

* The answers to practice problems can be found at the end of the section, just before the problems.

Otherwise, the sequence diverges, either because there is no one single number the terms approach or because the terms become unbounded.

Example 3

To what value, if any, does the sequence defined by $a_n = (-1)^n$ converge?

Solution

When in doubt, write out a few terms. The terms of this sequence are (with n starting at 0): 1, -1, 1, -1,…. The values of a_n bounce between -1 and 1. Since the sequence oscillates between these two values, it does not approach either. This sequence diverges. ◊

Example 4

Suppose s and t are positive numbers. Determine whether the sequence $a_n = \left(\frac{s}{t}\right)^n$ converges if …

a. $s < t$ b. $s > t$ c. $s = t$

Solution

a. If $s < t$, then $\frac{s}{t}$ is a number between 0 and 1. (Remember that s and t are both positive.) When we raise a number between 0 and 1 to higher and higher powers, the values we obtain get closer and closer to zero. In this case, the sequence converges to zero. If we further assume that s and t are greater than 1, we can take a different perspective. $a_n = \frac{s^n}{t^n}$. Since $t > s$, the denominator will grow much faster than the numerator as n increases. Since the denominator dominates the fraction, the fraction will go to zero. (Why was it important to assume that s and t are greater than 1 for this explanation?)

b. The situation here is the opposite of that in part (a). Now s/t is greater than 1, so the terms of the sequence will grow without bound as n increases. The sequence diverges.

c. If $s = t$, $\left(\frac{s}{t}\right)^n = 1^n = 1$ for all n. In this case the sequence converges to 1. ◊

Example 5

To what value, if any, does the sequence defined by $a_n = \left(1 + \dfrac{1}{n}\right)^n$ converge?

Solution

You might recognize the limit of this sequence as the definition of the number e. If so, you are right to answer that the sequence converges to e. If you did not recognize this, you could go two ways. A mathematically rigorous approach would be to look at the analogous function $f(x) = \left(1 + \frac{1}{x}\right)^x$ and take its limit as $x \to \infty$. This will involve using l'Hôpital's Rule as well as a few other tricks. It is a good exercise, and the replacement of a discrete object like a sequence with a corresponding continuous function will come up again in this chapter. However, we can also look at a table of values.

n	1	10	100	1,000	10,000	100,000
$\left(1 + \frac{1}{n}\right)^n$	2	2.59374	2.70481	2.71692	2.71814	2.71828

This provides pretty compelling evidence that the sequence converges to e. If you still do not recognize the number 2.71828… as being e, the best you can say is, "The sequence appears to converge to about 2.718." In fact, this is often good enough for the work we will be doing. ◊

Practice 4

Look back at the sequences in Examples 1 and 2 and Practices 1 and 2. Which of the sequences converge? To what values do they converge?

Series

For the purposes of this chapter, we actually don't care much about sequences. We care about infinite series (which we will usually abbreviate to just 'series'). However, as we will see in a moment, we need an understanding of *sequence* convergence to define *series* convergence. A series is a lot like a sequence, but instead of just listing the terms, we add them together. For example, the sequence

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$$

has corresponding series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots. \tag{1.1}$$

Writing the terms of a series can be cumbersome, so we often use sigma notation as a shorthand. In this way, series (1.1) can be written in the more concise form $\sum_{n=1}^{\infty}\frac{1}{n}$. The Σ is the Greek letter S (for sum), the subscript tells the index variable and its initial value. The superscript tells us what the "last" value of the index variable is (in the case of an infinite series, this will always be ∞). Finally, an expression for the general term of the series (equivalently, the parent *sequence*) follows the Σ.

What we really care about—in fact the major question driving the entire second half of this chapter—is whether or not a particular series converges. In other words, when we keep adding up all the terms in the series does that sum approach a particular value? This is a different kind of addition from what you have been doing since first grade. We are now adding together infinitely many things and asking whether it is possible to get a finite answer. If your intuition tells you that such a thing can never be possible, go back and review integration, particularly improper integration. (You'll probably need to review what you know about improper integrals for Section 7 anyway, so you can get a head start now if you like.)

There is no way we can actually add infinitely many terms together. It would take too long, among other technical difficulties. We need to get a sense of how a series behaves by some other method, and the way to do that is to examine the **partial sums** of the series. Going back to series (1.1), instead of adding *all* the terms, we can add the first few. For example, the 10^{th} partial sum of the series is obtained by adding together the first 10 terms:

$$s_{10}=\sum_{n=1}^{10}\frac{1}{n}.$$

We often (though not always or exclusively) use s_n to denote the n^{th} partial sum of a series.

Example 6

Evaluate the fifth, tenth, and twenty-fifth partial sums of the series $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$. Make a conjecture about $\lim_{b\to\infty}\sum_{n=0}^{b}\left(\frac{1}{2}\right)^n$.

Solution

We want to find s_5, s_{10} and s_{25}. (The counting is a little strange here. s_5 is the fifth partial sum in the sense that it adds up the terms through a_5, and that's what we mean by 'fifth' in this context. However, it does not add up five terms because of the zeroth term. The partial sum that adds up just a single term—the initial term—would be s_0.) Technology is a useful tool for computing partial sums. If you don't know how to make your calculator quickly compute these partial sums, ask your teacher for help.

$$s_5 = \sum_{n=0}^{5}\left(\frac{1}{2}\right)^n = 1.96875 \qquad s_{10} = \sum_{n=0}^{10}\left(\frac{1}{2}\right)^n \approx 1.9990 \qquad s_{25} = \sum_{n=0}^{25}\left(\frac{1}{2}\right)^n \approx 1.99999997$$

It would seem that the partial sums are approaching 2, so we conjecture that

$$\lim_{b\to\infty}\sum_{n=0}^{b}\left(\frac{1}{2}\right)^n = 2.$$

As we will see shortly, this conjecture is in fact correct. ◊

The previous example shows the way to think about convergence of series. Look at the partial sums s_n. These partial sums form a sequence of their own, and we already know how to talk about convergence for sequences. Here is a short list of the partial sums for the series in Example 6, this time presented as a sequence.

$$\begin{aligned} s_0 &= 1 \\ s_1 &= 1.5 \\ s_2 &= 1.75 \\ s_3 &= 1.875 \\ s_4 &= 1.9375 \\ s_5 &= 1.96875 \\ s_6 &= 1.984375 \\ &\vdots \end{aligned}$$

As the solution to Example 6 indicates, the key idea is to look at the sequence of partial sums and in particular at its limit as $n \to \infty$. If $\lim_{n\to\infty} s_n$ exists—that is, if the sequence of partial sums $\{s_n\}$ exists—then we say that the *series* Σa_n converges. The $\lim_{b\to\infty}\sum_{n=0}^{b}\left(\frac{1}{2}\right)^n$ which appeared in the solution to Example 6 is just a way of conceptualizing $\lim_{n\to\infty} s_n$.

All this is summarized in the following definition.

Definition A series **converges** if its sequence of partial sums converges. That is, we say that $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} s_n$ converges, where $s_n = a_1 + a_2 + a_3 + \cdots + a_n$. If the sequence of partial sums does not converge, we say that the series **diverges**. If the series converges, the limit of s_n is called the sum or value of the series.

Note, however, that the initial term of s_n may be a_0 instead of a_1. Actually, it could be a-sub-anything. It depends on how the series is defined. The important idea of series convergence—the question of what happens to the sequence of partial sums in the long run—is unaffected by where we start adding. The particular *value* of the series will certainly depend on where the index variable starts, of course, but the series will either converge or diverge independent of that initial index value. Sometimes we will simply write Σa_n when we do not want to be distracted by the initial value of the index.

Practice 5

For the series $\sum_{n=1}^{\infty}\frac{1}{3^n}$, list the first five terms of the series and the first five partial sums of the series. Does the series appear to converge?

The following theorem states three basic facts about working with convergent series.

Theorem 1.1

If $\sum_{n=1}^{\infty} a_n$ converges to A and $\sum_{n=1}^{\infty} b_n$ converges to B, then…

1. $\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n = cA$

2. $\sum_{n=1}^{\infty} (a_n \pm b_n) = A \pm B$

3. for any positive integer k, $\sum_{n=k}^{\infty} a_n$ converges, though almost certainly not to A.

The first part of Theorem 1.1 says that we can factor a common multiple out of all the addition, and the second says that we can split up a summation over addition. These are in keeping with our regular notions of how addition works since the Σ just denotes a *lot* of addition. The third statement tells us that the starting value of the index variable is irrelevant for determining whether a series converges. While the starting point does affect the *value* of the series, it will not affect the question of convergence. In other words, the first few terms (where 'few' can mean anywhere from 2 or 3 to several million) don't affect the convergence of the series. Only the long-term behavior of the series as $n \to \infty$ matters.

Practice 6

Compute several partial sums for the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Does the series seem to converge? Approximately to what?

Example 7

Compute several partial sums for the series $\sum_{n=1}^{\infty} \sqrt{n}$. Does the series seem to converge?

Solution

Let's explore a few partial sums.

n	1	2	3	4	5
a_n	1	1.414	1.732	2	2.236
s_n	1	2.414	4.146	6.146	8.382

It should not take long for you to convince yourself that this series will diverge. Not only do the partial sums get larger with every additional term, they do so at an increasing rate. If this behavior keeps up, then there is no way the partial sums can approach a finite number; they must diverge to infinity. And the reason for this is made clear by the values of a_n. The terms of the series are themselves increasing and, from what we know of the square root function, will continue to increase. The series must diverge. ◊

Example 7 brings up an essential criterion for convergence of series. In order for a series to converge, the terms of the parent sequence have to decrease towards zero as n goes to infinity. If we continue adding together more and more terms, and if those terms do not go to zero, the partial sums will always grow (or shrink, if the terms are negative) out of control. Convergence is impossible in such a situation. We summarize this observation in the following theorem.

Theorem 1.2 – The n^{th} Term Test
In order for a series to converge, it is necessary that the parent sequence converges to zero. That is, given a series $\sum a_n$, if $\lim\limits_{n\to\infty} a_n \neq 0$, then the series diverges.

Hopefully, Theorem 1.2 makes intuitive sense. It may be surprising to you that the converse is *not* a true statement. That is, even if $a_n \to 0$, it may be the case that Σa_n diverges. The most famous example of this is the so-called **harmonic series***: $\sum_{n=1}^{\infty} \frac{1}{n}$.

Example 8

Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Solution

Most textbooks present a proof for the divergence of the harmonic series based on grouping successive terms. This particular proof goes back to the 14th century, so it is a true classic of mathematics. If it is in your primary textbook, you should read it. However, I prefer a different proof. Let $H_k = \sum_{n=1}^{k} \frac{1}{n}$ and consider the quantity $H_{2k} - H_k$.

$$\begin{aligned} H_{2k} - H_k &= \left[\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{2k}\right] - \left[\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k}\right] \\ &= \frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k} \end{aligned}$$

But in each of the terms $\dfrac{1}{k + \text{something}}$ the denominator is no more than $2k$. That means each fraction must be at least as large as $\dfrac{1}{2k}$. Hence we have

$$\begin{aligned} H_{2k} - H_k &= \frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k} \\ &\geq \underbrace{\frac{1}{2k} + \frac{1}{2k} + \frac{1}{2k} + \cdots + \frac{1}{2k}}_{k \text{ terms}} = \frac{k}{2k} = \frac{1}{2} \end{aligned}$$

Rearranging, $H_{2k} \geq H_k + \frac{1}{2}$. This means that no matter what the partial sum H_k is, if we go twice as far in the sequence of partial sums, we are guaranteed to add at least another 0.5. The series cannot possibly converge, because we can always find a way to increase any partial sum by at least 0.5. We conclude that the harmonic series diverges. ◊

* Where does the harmonic series get its name? If you pluck a string, it vibrates to produce a sound. The vibrations correspond to standing waves in the string. These waves must be such that half their wavelengths are equal to the length of the string, half the length of the string, a third the length of the string, etc. The tone produced from a standing wave whose wavelength is twice the length of the string is called the fundamental frequency. The additional tones are called overtones or, wait for it, harmonics. (Different instruments produce different harmonics for the same fundamental frequency. This is why a violin, for example, sounds different from a tuba even when playing the same note.) Mathematicians appropriated the term harmonic from acoustics and use it to describe things in which we see the fractions one, one half, one third, and so on.

To summarize: If the terms of a series do not go to zero as $n \to \infty$, then the series diverges. But if the terms do go to zero, that does not necessarily mean that the series will converge. The n^{th} term test cannot show convergence of a series. Most students incorrectly use the n^{th} term test to conclude that a series converges at least once in their career with series. If I were you, I would do that soon to get it over with so that you do not make the mistake again somewhere down the line.

Geometric Series

A very important class of series that you probably saw in precalculus is the **geometric series**. A geometric series is one in which successive terms always have the same ratio. This ratio is called the **common ratio**, and it is typically denoted *r*. For example, the following series are geometric with *r* equal to 2/3, 2, 1, and -1/2, respectively:

$$6+4+\frac{8}{3}+\frac{16}{9}+\frac{32}{27}+\cdots \qquad 1+2+4+8+16+\cdots$$

$$7+7+7+7+7+\cdots \qquad 10-5+\frac{5}{2}-\frac{5}{4}+\frac{5}{8}+\cdots$$

In general, the terms of a geometric series can be expressed as $a \cdot r^n$ where a is the initial term of the series and r is the common ratio. In the example of $1+2+4+\cdots$ with a = 1 and r = 2, every term has the form $1 \cdot 2^n$. n is the index of the term (starting at 0 in this example).

There are several reasons why geometric series are an important example to consider.

1. It is comparatively easy to determine whether they converge or diverge.
2. We can actually determine the *value* to which the series converges. (This is often a much harder task than just figuring out whether or not the series converges.)
3. They are good standards for comparison for many other series that are not geometric. This is one of the ways that geometric series will come up again and again during this chapter.

Practice 7

Look at partial sums for the series $\sum_{n=0}^{\infty}\left[16 \cdot 4^n\right]$ and $\sum_{n=0}^{\infty}\left[16 \cdot \left(\frac{1}{4}\right)^n\right]$.

Which seems to converge? To what?

You should have found that the first series in Practice 7 diverges, while the second one converges. The divergence of the first series should not have been a surprise; it doesn't pass the n^{th} term test. In fact, any geometric series with $r \geq 1$ or $r \leq -1$ will have to diverge for this reason. (We will typically group both inequalities of this form together and say something like: If $|r| \geq 1$, then the geometric series diverges by the n^{th} term test.) It is our good fortune with geometric series that if $|r| < 1$ the series will converge. Even more, as the next theorem tells us, we can compute the sum to which the series converges.

Theorem 1.3 – The Geometric Series Test

If $|r| < 1$, then the geometric series $\sum_{n=0}^{\infty} ar^n$ converges. In addition, the sum of the series is $\frac{a}{1-r}$. If $|r| \geq 1$ then the geometric series diverges. Note that a is the initial term of the series.

We won't give formal proofs for many of the theorems and convergence tests in this chapter. But since geometric series are so fundamental, it is worth taking a moment to give a proof.

Proof
We have already argued that a geometric series with $|r| \geq 1$ will diverge based on the n^{th} term test. All that remains is to show that when $|r| < 1$ the series converges and its sum is as given in the theorem.

The partial sums of the series $\sum_{n=0}^{\infty} ar^n$ have the form $s_n = a + ar + ar^2 + \cdots + ar^n$. It turns out that we can actually write an explicit formula for s_n. We often cannot do such a thing, so we might as well take advantage of the opportunity.

$$a + ar + ar^2 + \cdots + ar^n = a\frac{1 - r^{n+1}}{1 - r} \tag{1.2}$$

To verify that equation (1.2) is valid, simply multiply both sides by $(1 - r)$. After distributing and simplifying, you will find that the equality holds.

Now that we have an expression for the partial sum s_n, all we need to do is see if the sequence it generates converges. In other words, we examine $\lim_{n\to\infty} s_n$. Note that if $|r| < 1$ as hypothesized, then $\lim_{n\to\infty} r^{n+1} = 0$.

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^n \\ s_n &= a\frac{1 - r^{n+1}}{1 - r} \\ \lim_{n\to\infty} s_n &= \lim_{n\to\infty} a\frac{1 - r^{n+1}}{1 - r} \\ \lim_{n\to\infty} s_n &= a\frac{1}{1 - r} \end{aligned}$$

We see that the sequence of partial sums converges, so the series converges. Moreover, the sequence of partial sums converges to a particular value, $\frac{a}{1-r}$, so this is the sum of the series. ◊

Example 9
Determine if the following series converge. If they do, find their sums.

a. $\sum_{n=0}^{\infty} \frac{4}{3^n}$ b. $\sum_{n=0}^{\infty} 3 \cdot 2^n$ c. $\sum_{n=4}^{\infty} \left[3 \cdot \left(\frac{2}{3}\right)^n \right]$ d. $\sum_{n=0}^{\infty} \frac{3^{n+1}}{5^n}$

Solution

a. The general term a_n can be rewritten as $4 \cdot \frac{1}{3^n}$ or even $4 \cdot \left(\frac{1}{3}\right)^n$. This makes it clear that the series is geometric with $r = 1/3$. Since $|r| = \frac{1}{3} < 1$, the series converges to $\frac{4}{1 - \frac{1}{3}} = \frac{4}{\frac{2}{3}} = 6$.

b. Here, $|r| = 2 > 1$. The series diverges.

c. The third series is clearly geometry with $r = 2/3$. $|r| < 1$, so the series converges. It is tempting to say that the sum is $\frac{3}{1 - \frac{2}{3}}$ or 9. However, note the initial value of the index variable. In this case, I think it is useful to write out a few terms: $3\left(\frac{2}{3}\right)^4 + 3\left(\frac{2}{3}\right)^5 + 3\left(\frac{2}{3}\right)^6 + \cdots$. This helps clarify that the initial term is not 3, but $3\left(\frac{2}{3}\right)^4$ or 16/27. The a in $a/(1-r)$ stands for this initial term, so the sum should be $\frac{\frac{16}{27}}{1-\frac{2}{3}}=\frac{16}{9}$. Experimenting with partial sums should show you the reasonableness of this answer. Technically, though, this is not what Theorem 1.3 says we can do. A more rigorous approach is the following:

$$\begin{aligned}\sum_{n=4}^{\infty}3\cdot\left(\tfrac{2}{3}\right)^n &= 3\left(\tfrac{2}{3}\right)^4+3\left(\tfrac{2}{3}\right)^5+3\left(\tfrac{2}{3}\right)^6+\cdots\\ &= 3\left(\tfrac{2}{3}\right)^4\left[1+\tfrac{2}{3}+\left(\tfrac{2}{3}\right)^2+\cdots\right]\\ &= 3\left(\tfrac{2}{3}\right)^4\cdot\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n\end{aligned}$$

Theorem 1.3 *does* legitimately allow us to evaluate the sum in this last expression. When we do so and multiply by the $3\left(\frac{2}{3}\right)^4$ coefficient, we arrive at the same answer of 16/9.

d. At first glance, this doesn't look like a geometric series. Do not despair! We want the powers to be n, so let's use exponent properties to make that happen.

$$\sum_{n=0}^{\infty}\frac{3^{n+1}}{5^n}=\sum_{n=0}^{\infty}\frac{3\cdot 3^n}{5^n}=\sum_{n=0}^{\infty}3\cdot\left(\tfrac{3}{5}\right)^n$$

And now we see that we have a geometric series after all. Since $|r|=\frac{3}{5}<1$, the series converges, and its sum is $\frac{3}{1-\frac{3}{5}}=2.4$ If this did not occur to you, then try writing out some terms.

$$\sum_{n=0}^{\infty}\frac{3^{n+1}}{5^n}=\frac{3}{1}+\frac{9}{5}+\frac{27}{25}+\frac{81}{125}+\cdots$$

Now we see the initial term is indeed 3, while the common ratio is 3/5. ◊

There are two morals to take away from Example 8. First, when in doubt, write out a few terms. Often seeing the terms written in "expanded form" will help clarify your thinking about the series. The converse is also good advice. If you have a series in expanded form and you are unsure how to proceed, try writing it with sigma notation. The second moral is that we can apply Theorem 1.3 a little more broadly than indicated by the statement of the theorem. As long as a geometric series converges, its sum is given by $\frac{\text{initial term}}{1-\text{common ratio}}$. You can prove this if you like; it will be a corollary to Theorem 1.3.

Example 10

Write 0.2323232323… as a fraction.

Solution

$0.2323232323\ldots=0.23+0.0023+0.000023+0.00000023+\cdots$

This is a geometric series with initial term 0.23 and common ratio 1/100. Therefore its sum is $\frac{0.23}{1-\frac{1}{100}}=\frac{0.23}{0.99}=\frac{23}{99}$. And there you have it. In essentially the same way, *any* repeating decimal can be turned into a fraction of integers. ◊

Closing Thoughts

Conceptually, when we are trying to decide the convergence of a series we are looking at how quickly $a_n \to 0$. The n^{th} term test tells us that we need to have $\lim_{n\to\infty} a_n = 0$ in order to have any hope of convergence, but the example of the harmonic series shows us that just having a_n tend to zero is insufficient. Yes, the terms of the harmonic series get small, but they do not do so fast enough for the sum of all of them to be finite. There's still too much being added together. For a geometric series with $|r| < 1$, on the other hand, the terms go to zero *fast*. They quickly become so small as to be insignificant, and that is what allows a geometric series to converge. (Compare these ideas to what you studied with improper integrals.) As we continue to study criteria for the convergence of series later in this chapter, this idea of how quickly the terms go to zero will be a good perspective to keep in mind.

I will close this section with one more series:

$$1-1+1-1+1-1+1-1+\cdots. \tag{1.3}$$

The question, as always, is does series (1.3) converge or diverge? Answer this question before reading on.

It turns out that this is a tough question for students beginning their study of infinite series because our intuition often gives us an answer that is at odds with the "right" answer. Some students say that successive terms will clearly cancel, so the series should converge to 0. Other students will look at the partial sums. Convince yourself that s_n is equal to either 0 or 1, depending on the parity (oddness or evenness) of n. Some therefore say that the series converges to 0 *and* 1. Others suggest splitting the difference, saying that the series converges to 1/2. None of these answers is correct, and none takes into account the definition of series convergence. Since the partial sums do not approach a single number, the series diverges. Or, if you prefer, this is a geometric series with $r = -1$, and $|-1|$ is not less than 1. This series diverges by the geometric series test. In fact, like all divergent geometric series, it can be shown to diverge by the n^{th} term test; $\lim_{n\to\infty} a_n$ does not exist since the terms oscillate between 1 and -1. If the limit does not exist, it is definitely not zero.

If you arrived at one of these incorrect answers, don't feel too bad. When series were a relatively new concept in the 17^{th} and 18^{th} centuries, many famous mathematicians came to these incorrect answers as well. Here is a more sophisticated argument for the sum being 1/2.

Suppose that the series converges and call its sum S.

$$S = 1-1+1-1+1-1+\cdots$$

Now let us group all the terms but the first.

$$\begin{aligned} S &= 1-1+1-1+1-1+\cdots \\ &= 1-(1-1+1-1+\cdots) \\ &= 1-S \end{aligned}$$

Since $S = 1 - S$, some quick algebra shows us that S must equal 1/2.

The problem with this argument is that we have assumed that the series converges when in fact it does not. Once we make this initial flawed assumption, all bets are off; everything that follows is a fallacy. This is a cautionary tale against rashly applying simple algebra to divergent series. This theme will be picked up and expanded in Section 9.

Leibniz, one of the inventors of calculus, also believed that the series should be considered as converging to 1/2, but for different reasons. His argument was probabilistic. Pick a partial sum at random. Since you have an equally likely change of picking a partial sum equal to 0 or 1, he thought the average of 1/2 should be considered the sum.

I lied. *One* more series. The series

$$1+2+4+8+16+\cdots$$

is geometric with $r = 2$. Euler, one of the greatest mathematicians of all time, said that the series therefore converges to $\frac{1}{1-2}$: the initial term over 1 minus the common ratio. Now it was well known to Euler that $1+2+3+4+\cdots=\infty$. He made the following string of arguments.
First,

$$\frac{1}{1-2}=1+2+4+8+16+\cdots$$

because that's the formula for geometric series. Second,

$$1+2+4+8+\cdots>1+2+3+4+\cdots$$

because each term of the left-hand series is at least as large as the corresponding term of the right-hand series. Third, since $1+2+3+4+\cdots=\infty$,

$$\frac{1}{1-2}>\infty .$$

In other words, $-1>\infty$.

Now Euler was no fool; in addition to making huge advances in the development of existing mathematics, he created entirely new fields of math as well. But in his time this issue of convergence was not well understood by the mathematical community. (Neither were negative numbers, actually, as Euler's argument shows. Does it surprise you that as late as the 18th century mathematicians did not have a complete handle on negative numbers and sometimes viewed them with suspicion?)*

My point with these last couple series is not to try to convince you that series (1.3) converges (it doesn't) or that $-1>\infty$ (it isn't). My point is that historically when mathematicians were first starting to wrestle with some of the concepts that we have seen in this section and will continue to see for the rest of the chapter, they made mistakes. They got stuff wrong sometimes. Much of what we will be studying may strike you as counterintuitive or bizarre. Some of it will be hard. But keep at it. If the likes of Leibniz and Euler were allowed to make mistakes, then surely you can be permitted to as well. But give it your best shot. Some of the stuff you will learn is pretty amazing, so it is well worth the struggle.

Answers to Practice Problems

1. $\frac{1}{0!},\frac{1}{1!},\frac{1}{2!},\frac{1}{3!},\frac{1}{4!}$ or $1,1,\frac{1}{2},\frac{1}{6},\frac{1}{24}$

2. If we start with $n = 1$, then we have $a_n=\frac{n^3}{n+1}$.

3. $a_n=\frac{n}{2n-1}$

4. $a_n=\frac{(-1)^n}{n^2}$ converges to 0 because the denominator blows up. $a_n=\frac{n}{2n-1}$ converges to 1/2. If you need convincing of this, consider $\lim\limits_{x\to\infty}\frac{x}{2x-1}$. $a_n=\frac{1}{n!}$ converges to 0. $a_n=\frac{n^3}{n+1}$ diverges.

* The ideas in this discussion of some historical aspects surrounding series convergence are discussed at length in Morris Kline's *Mathematics: The Loss of Certainty*.

5. In the series $\sum_{n=1}^{\infty} \frac{1}{3^n}$, we have $a_n = \frac{1}{3^n}$. The first five terms of the series are therefore $\frac{1}{3^1}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{3^4},$ and $\frac{1}{3^5}$. (If you prefer: $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}$.) The first five partial sums are as follows.

$$\begin{aligned}
s_1 &= \tfrac{1}{3} &&= \tfrac{1}{3} \\
s_2 &= \tfrac{1}{3} + \tfrac{1}{9} &&= \tfrac{4}{9} \\
s_3 &= \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} &&= \tfrac{13}{27} \\
s_4 &= \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \tfrac{1}{81} &&= \tfrac{40}{81} \\
s_5 &= \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \tfrac{1}{81} + \tfrac{1}{243} &&= \tfrac{121}{243}
\end{aligned}$$

It appears that these partial sums are converging to 1/2, so the series seems to converge.

6. Some sample partial sums: $s_{10} \approx 1.5498$, $s_{50} \approx 1.6251$, $s_{100} \approx 1.6350$. The series appears to converge to roughly 1.64. In fact, the series converges to $\pi^2/6$, something proved by Euler.

7. $\sum_{n=0}^{\infty} \left[16 \cdot 4^n\right]$ definitely diverges. It does not pass the n^{th} term test since $\lim_{n\to\infty} 16 \cdot 4^n$ does not exist.

$\sum_{n=0}^{\infty} \left[16 \cdot \left(\frac{1}{4}\right)^n\right]$ converges. $s_{10} \approx 21.333328$ and s_{50} agrees with $21\frac{1}{3}$ to all decimal places displayed by my calculator. In fact, $21\frac{1}{3}$ is the value of the series.

Section 1 Problems

1. Write out the first 5 terms of the sequences defined as follows. Let the initial value of n be zero unless there is a reason why it cannot be. Simplify as much as possible.

   a. $a_n = \dfrac{(n+1)!}{n!}$

   b. $a_n = \dfrac{\cos(n\pi)}{n}$

   c. $a_n = \sqrt[n]{n}$

   d. $a_n = \ln(\ln n)$

   e. $a_n = \dfrac{3^n}{4^{n+1}}$

2. Give an expression for the general term of the following sequences.

   a. $2, 4, 6, 8, \ldots$

   b. $-1, 1, \dfrac{-1}{2}, \dfrac{1}{6}, \dfrac{-1}{24}, \dfrac{1}{120}, \cdots$

   c. $1, 4, 27, 256, \ldots$

3. Which of the sequences in Problems 1 and 2 converge? To what value do they converge?

4. For what values of x does the sequence defined by $a_n = x^n$ converge?

5. For what values of x does the sequence defined by $a_n = \dfrac{x^n}{n!}$ converge? To what value does it converge?

6. Evaluate s_5 and s_{10} for the following series. If you can, make a conjecture for the sum of the series.

   a. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$

   b. $3 - 1 + \frac{1}{3} - \frac{1}{9} + \cdots$

   c. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

In problems 7-16, determine whether the given series definitely converges, definitely diverges, or its convergence cannot be determined based on information from this section. Give a reason to support your answer.

7. $1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots$

8. $2 - 4 + 8 - 16 + 32 - \cdots$

9. $\dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} + \dfrac{4}{5} + \cdots$

10. $\displaystyle\sum_{n=1}^{\infty} \cos\left(\tfrac{1}{n}\right)$

11. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\tfrac{1}{n}\right)$

12. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$

13. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{\pi}{e}\right)^n$

14. $\displaystyle\sum_{n=1}^{\infty} \sin(n)$

15. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{3^n + n}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

In problems 17-24, find the sum of the convergent series.

17. $\displaystyle\sum_{n=0}^{\infty} \left(\tfrac{2}{5}\right)^n$

18. $\displaystyle\sum_{n=0}^{\infty} 2 \cdot \left(\tfrac{1}{8}\right)^n$

19. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{8^{n+2}}$

20. $\displaystyle\sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n}$

21. $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^n}$

22. $\displaystyle\sum_{n=10}^{\infty} \left(\tfrac{3}{4}\right)^n$

23. $\displaystyle\sum_{n=1}^{\infty} \tfrac{5}{3^n}$

24. $\displaystyle\sum_{n=0}^{\infty} \frac{5 + 3^n}{4^n}$

25. Represent the following repeating decimals as fractions of integers.

    a. $0.777\overline{7}$ c. $0.317\overline{317}$

    b. $0.8\overline{2}$ d. $2.438\overline{38}$

26. Express $0.\overline{9}$ as a geometric series. Use this representation to prove that $0.\overline{9}=1$.

27. A superball is dropped from a height of 6 ft and allowed to bounce until coming to rest. On each bounce, the ball rebounds to 4/5 of its previous height. Find the total up-and-down distance traveled by the ball.

28. Repeat exercise 27 for a tennis ball that rebounds to 1/3 of its previous height after every bounce. This time suppose the ball is dropped from an initial height of 1 meter.

29. Repeat exercise 27 for a bowling ball dropped from 2 feet that rebounds to 1/100 of its previous height after every bounce.

30. The St. Ives nursery rhyme goes as follows:

    "As I was walking to St. Ives / I met a man with 7 wives / Each wife had seven sacks / Each sack had seven cats / Each cat had seven kits / Kits, cats, sacks, wives / How many were going to St. Ives?"

    Use sigma notation to express the number of people and things (kits, cats, etc.) that the narrator encountered. Evaluate the sum.

31. Evaluate the following sum:

    $$\sum_{\substack{k \text{ is divisible} \\ \text{by 2 or 3}}}^{\infty} \frac{1}{k} = \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{12}\cdots$$

    (Hint: This series can be regrouped as an infinite series of geometric series.)

32. Evaluate the sum $\sum_{n=1}^{\infty}\frac{n}{2^n}$.

    (Hint: Rewrite all the fractions as unit fractions (for example, rewrite the term $\frac{3}{2^3}$ as $\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$). Then regroup to form an infinite series of geometric series.)

33. Generalize the result of Problem 32 to give the sum of $\sum_{n=1}^{\infty}\frac{n}{r^n}$, where $|r|>1$.

34. Another type of series is the so-called "telescoping series." An example is

    $$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots$$
    $$=\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cancel{\cdots}.$$

    This series is called a "telescoping" series because it collapses on itself like a mariner's telescope.

    a. Find an expression for the partial sum s_n of the telescoping series shown above.

    b. Compute the sum of this telescoping series.

In problems 35-38, find the sums of the telescoping series or determine that they diverge. You may have to use some algebraic tricks to express the series as telescoping.

35. $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)$

36. $\sum_{n=2}^{\infty}\frac{4}{n^2-1}$

37. $\sum_{n=1}^{\infty}\left(\frac{3}{\sqrt{n}}-\frac{3}{\sqrt{n+1}}\right)$

38. $\sum_{n=1}^{\infty}\frac{1}{n^2+n}$

39. $\sum_{n=2}^{\infty}\ln\left(\frac{n}{n+1}\right)$

40. $\sum_{n=0}^{\infty}\left(\arctan(n+1)-\arctan(n)\right)$

41. Give an example of two *divergent* series $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ such that $\sum_{n=1}^{\infty}\frac{a_n}{b_n}$ converges.

For problems 42-47 indicate whether the statement is True or False. Support your answer with reasons and/or counterexamples.

42. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

43. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

44. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} s_n = 0$.

45. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges.

46. If $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges, then $\sum_{n=1}^{\infty} a_n$ converges.

47. If a telescoping series of the form $\sum_{n=1}^{\infty} (a_n - b_n)$ converges, then $\lim_{n\to\infty} b_n = 0$.

48. The Koch Snowflake, named for Helge von Koch (1870-1924), is formed by starting with an equilateral triangle. On the middle third of each side, build a new equilateral triangle, pointing outwards, and erase the base that was contained in the previous triangle. Continue this process forever. The first few "stages" of the Koch Snowflake are shown in Figure 1.1.

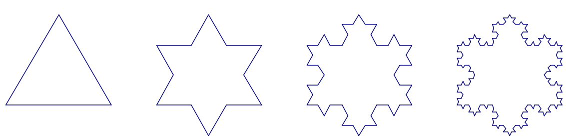


Figure 1.1: Stages of the Koch Snowflake

Call the equilateral triangle "stage 0," and assume that the sides each have length 1.

a. Express the perimeter of the snowflake at stage *n* as a geometric sequence. Does this sequence converge or diverge? If it converges, to what?

b. Express the area bounded by the snowflake as a geometric series. Does this series converge or diverge? If it converges, to what? (Hint: The common ratio is only constant <u>after</u> stage 0. So you will need to take that into account in summing your series.)

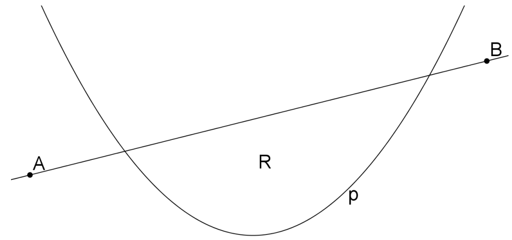


Figure 1.2: Region *R* is the parabolic sector bounded by parabola *p* and line $\overleftrightarrow{AB}$.

49. In the third century BCE, Archimedes developed a method for finding the area of a parabolic sector like the one shown in Figure 1.2.

Archimedes' method was as follows. First, he found the point at which the line tangent to the parabola was parallel to the secant line. (This was millennia before the MVT was articulated.) In the figures to follow (next page), this point is called *M*.

He then connected point *M* to the points at the end of the segment. This produced a triangle, whose area we will call *T*, as well as two new parabolic sectors: one cut by $\overline{MC}$ and the other cut by $\overline{MD}$. (See Figure 1.3 on the next page.)

He repeated the process with these new parabolic sectors to obtain points *U* and *V*. Archimedes then showed that the new triangles each had 1/8 the area of the original triangle. That is, each one had area *T*/8.

Now Archimedes simply repeated. Every parabolic sector was replaced with a triangle and two new sectors, and each triangle had 1/8 the area of the triangle that preceded it. He continued the process indefinitely to fill the original parabolic sector with triangles (Figures 1.4 and 1.5). This strategy is known as the method of exhaustion.

a. Use summation notation to express the area of the original parabolic sector in Figures 1.3-1.5. Your answer will be in terms of *T*.

b. Evaluate the sum from part (a). This was the result Archimedes derived in his "Quadrature of the Parabola."

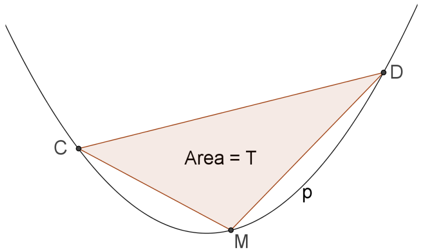


Figure 1.3: The first triangle

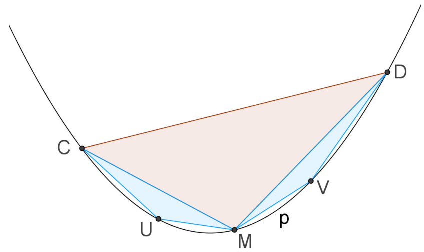


Figure 1.4: New triangles 1/8 the area of the original triangle

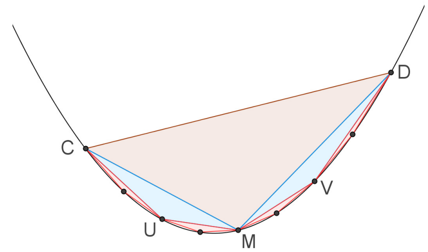


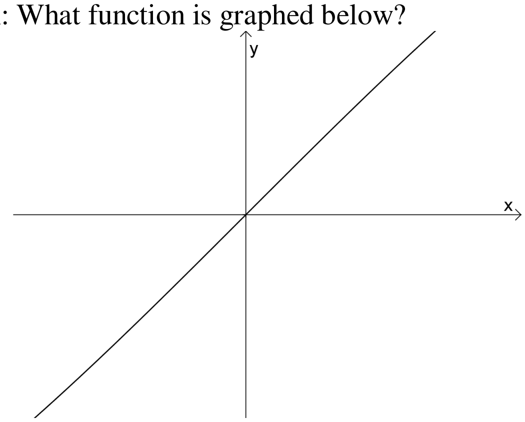
Figure 1.5: More and more triangles to fill the region

Section 2 – An Introduction to Taylor Polynomials

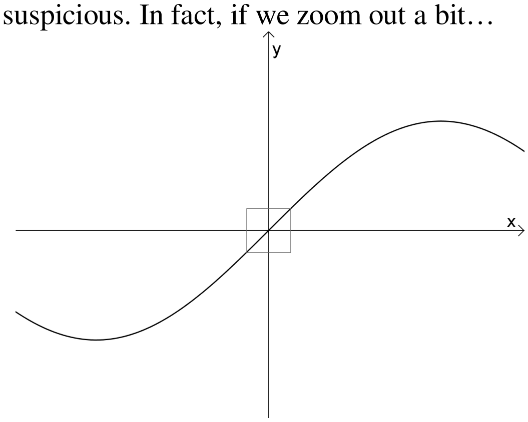
In this and the following few sections we will be exploring the topic of Taylor polynomials. The basic idea is that polynomials are easy functions to work with. They have simple domains, and it is easy to find their derivatives and antiderivatives. Perhaps most important, they are easy to evaluate. In order to evaluate a polynomial function like $f(x) = 3 + 2x - x^2 + \frac{1}{7}x^3$ at some particular x-value, all we have to do is several additions and multiplications. (Even the exponents are just shorthand for repeated multiplication.) I guess there's some subtraction and division as well, but you can view subtraction and division as being special cases of addition and multiplication. The point is that these are basic operations that we have all been doing since third grade. If we want to evaluate the function f above at $x = 2.3$, it will be a little tedious to do it by hand, but we *can* do it by hand if we choose. Another way to think about this is to say that we really know what $f(2.3)$ means; anything we can compute by hand is something that we understand fairly well.

Compare evaluating a polynomial to trying to evaluate $\cos(8)$, $e^{7.3}$, or even $\sqrt{10}$. Without a calculator, these are difficult expressions to approximate; we don't know how to compute these things. The functions $g(x) = \cos(x)$, $h(x) = e^x$, and $k(x) = \sqrt{x}$ are not functions that we can evaluate easily or accurately, except perhaps at a few special x-values. This is where Taylor polynomials come in. A Taylor polynomial is a polynomial function that we use in place of the "hard" functions like g, h, and k. Building, analyzing, and using these polynomials will occupy us for the next three sections.

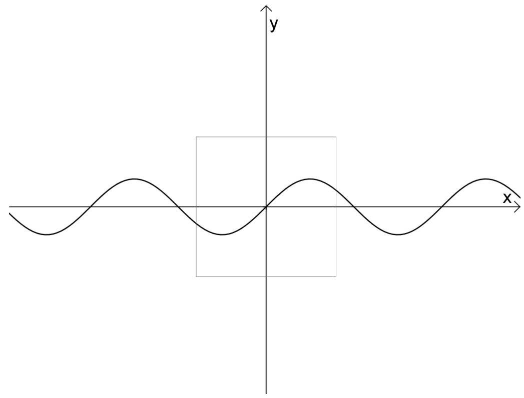
We begin with a question: What function is graphed below?



If you said the graph is of $y = x$, you said exactly what you were supposed to say. However, you might have found the lack of a scale suspicious. In fact, if we zoom out a bit…



… we see that we've been tricked by local linearity. Initially we were looking at the part of the graph in the box, and at that small scale it appeared to be a line. In this larger view, we see that this is actually the graph of a cubic polynomial. Or is it? Zoom out some more…



… and we see that actually this was the graph of our old friend the sine function all along. Again, the box shows what our viewing window was in the previous figure.

The point of this exercise was not actually to trick you, but to discover something new about differentiable functions. We already knew that $y = x$ approximates the sine function near the origin. (If you are not comfortable with this statement, go back and review linearization.) We also know that this approximation breaks down as we move away from $x = 0$. What the graphs above suggest is that there might be a cubic function that does a better job of modeling the sine function on a wider interval. Put down this book for a minute and play around with your graphing calculator. Can you find a cubic function that works well for approximating $\sin(x)$?

Modeling the Sine Function

I will present a rough method for building a polynomial model of the sine function one term at a time. We already know that $\sin(x) \approx x$ for small values of x. So let's hang on to that and tack on a cubic term. The cubic term will have to be of the form $-kx^3$. For one thing, near the origin the graph of the sine function looks like a "negative cubic," not a positive. For another, we can see from graphing the sine function along with $y = x$ that $x > \sin(x)$ for positive x-values. Thus we need to subtract something from our starting model of $y = x$. (And for negative x-values, we have the opposite: $x < \sin(x)$. This means we need to add to x, and $-kx^3$ will do that for $x < 0$.)

So let's start simple and tack on a $-x^3$ term. Unfortunately, Figure 2.1 (next page) shows that $y = x - x^3$ is no good. The graph is *too* cubic; the bend takes over at x-values that are too close to zero, and we end up with a graph that approximates the sine graph worse than our initial linear model. So we need to reduce the influence of the cubic term by using a coefficient closer to zero. At random and because 10 is a nice round number, let's try $y = x - \frac{1}{10}x^3$. In Figure 2.2 we see that we have overdone it. Now the cubic character of the curve takes over too late, and we still haven't gained much over the linear estimate. Try adjusting the coefficient of the cubic term until you get a polynomial that seems to fit well.

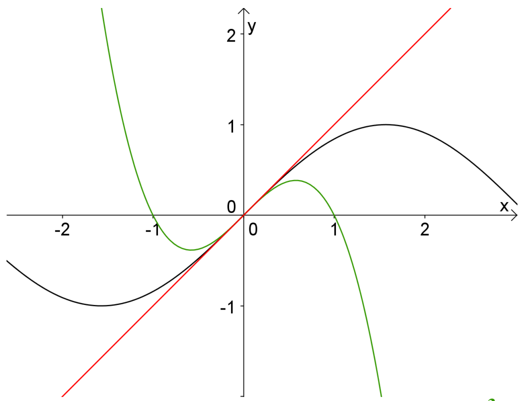


Figure 2.1: $y = \sin(x)$, $y = x$, and $y = x - x^3$

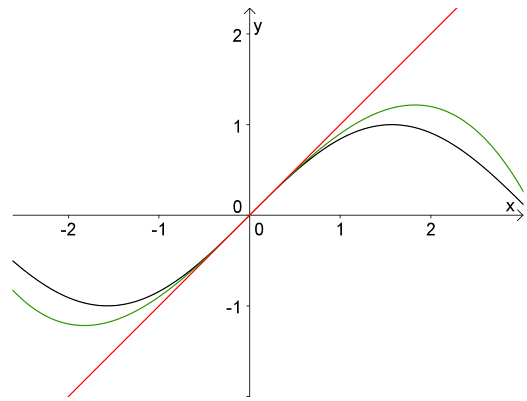


Figure 2.2: $y = \sin(x)$, $y = x$, and $y = x - \frac{1}{10}x^3$

After some trial-and-error, I think that $y = x - \frac{1}{6}x^3$ does a pretty good job of capturing the cubic character of the sine function near the origin. Graphically, this function seems to match the sine function pretty well on a larger interval than the linear approximation of $y = x$ (Figure 2.3). A table of values concurs. (Negative values have been omitted not because they are unimportant but because all the functions under consideration are odd; the y-values for corresponding negative x are just the opposite of the positive values shown.) We see that, while the values of x alone approximate the values of the sine function reasonably well for $|x| < 0.4$, the cubic expression approximates the values of the sine function very well until starting to break down around $|x| = 1$.

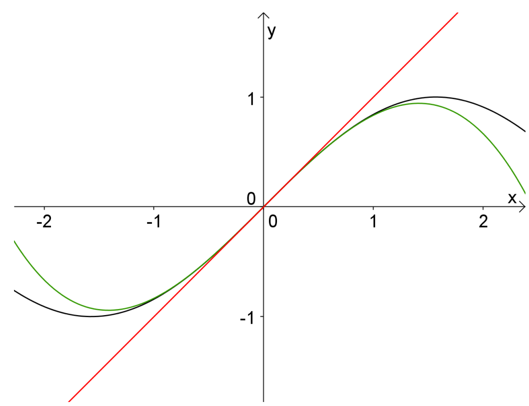


Figure 2.3: $y = \sin(x)$, $y = x$, $y = x - \frac{1}{6}x^3$

x	0	0.2	0.4	0.6	0.8	1.0	1.2
$y = \sin(x)$	0	0.1987	0.3894	0.5646	0.7174	0.8415	0.9320
$y = x - \frac{1}{6}x^3$	0	0.1987	0.3983	0.5640	0.7147	0.8333	0.9120

Can we do even better? The additional "bumpiness" of the sine function suggests that there is something to be gained from adding yet higher-order terms to our polynomial model. See if you can find an even better model. Really. Put down this book, pick up your calculator, and try to get a higher-order polynomial approximation for the sine function. I will wait.

Okay. Good work. Unfortunately, you can't tell me what you came up with, but I will share with you what I found. I started with what we had so far and then skipped right to a quintic term. It looked like adding in $\frac{1}{50}x^5$ was too much quintic while $\frac{1}{200}x^5$ seemed like too little. I think somewhere around $\frac{1}{120}x^5$ seems to be a good approximation as shown graphically (Figure 2.4) and numerically (below). The exact value of the coefficient doesn't matter too much at present. We will worry about getting the number right in Section 3. For now, it is the *idea* that we can find some polynomial function that approximates the sine curve that is important. In any event, look at the graph and the values in the table. The polynomial $y = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$ matches the sine function to *four decimal places* for x in the interval $-0.7 < x < 0.7$. To two decimal places, this quintic is good on the interval $-1.6 < x < 1.6$. It is a tremendous match for a relatively simple function.

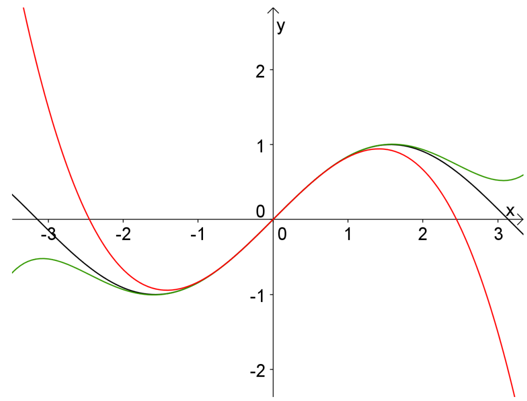


Figure 2.4: $y = \sin(x)$, $y = x - \frac{1}{6}x^3$, **and** $y = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

x	0.7	0.9	1.1	1.3	1.5	1.7	1.9	2.1
$y = \sin(x)$	0.6442	0.7833	0.8912	0.9636	0.9975	0.9917	0.9463	0.8632
$y = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$	0.6442	0.7834	0.8916	0.9648	1.0008	0.9995	0.9632	0.8968

Of course, there's no real reason to stop at the fifth-degree term. I'll leave the guessing-and-checking to you, but here's my seventh-degree polynomial approximation of the sine function:

$\sin x \approx x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \dfrac{x^7}{5040}$. This is graphed in Figure 2.5. As you can see, we now have a high quality approximation on what looks like the interval $-3 < x < 3$. (If you look at a table of values, the quality of the fit isn't actually all that great at $|x| = 3$. But definitely in, say, $-2.6 < x < 2.6$ this seventh-degree polynomial is an excellent approximation for the sine function.) One question worth asking, though, is what are the criteria for calling the approximation "good"? We will revisit this question in Section 4.

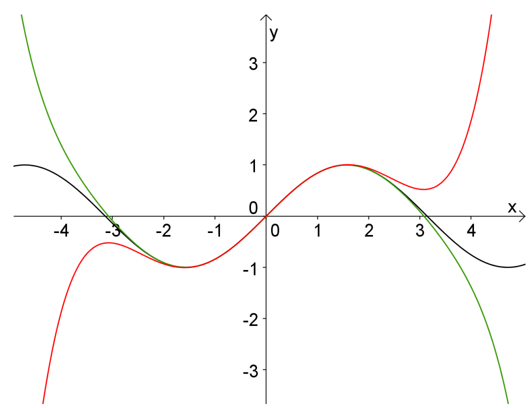


Figure 2.5: The sine function with quintic and seventh-degree polynomial approximations

Another question you might be asking is where the denominator of 5040 came from. It seems more than a little arbitrary. If you are asking whether it is really any better than using 4900, 5116, or some other random number in that neighborhood, then you are asking an excellent question. It turns out, though, that 5040 has a nice, relatively simple relationship to 7, the power of that term in the polynomial. Furthermore, it is the same relationship that 120 (the denominator of the fifth-degree term) has to 5, that 6 has to 3, and even that 1 (the denominator of the linear term) has to 1. The denominators I have chosen are the factorials of the corresponding powers; they are just (power)!. (I'm very excited about this, so I would like to end the sentence with an exclamation point. But then it would read "… (power)!!" which is too confusing. Later on, we will meet the double factorial function that actually is notated with two exclamation points.) In the next section we'll see *why* using the factorials is almost inevitable, but for now let's capitalize on the happenstance and see if we can generalize the pattern.

It seems that we have terms that alternate in sign, use odd powers, and have coefficients of the form 1/(power!). In other words,

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \tag{2.1}$$

Graphs of these polynomials for several values of *n* are shown in Figure 2.6 along with the graph of $y = \sin x$. (Note: These are values of *n*, not values of the highest power in the polynomial. The degree of the polynomial, based on Equation (2.1), is $2n + 1$. This means that $n = 8$ corresponds to a $(2 \cdot 8 + 1)^{\text{th}} = 17^{\text{th}}$-degree polynomial.)

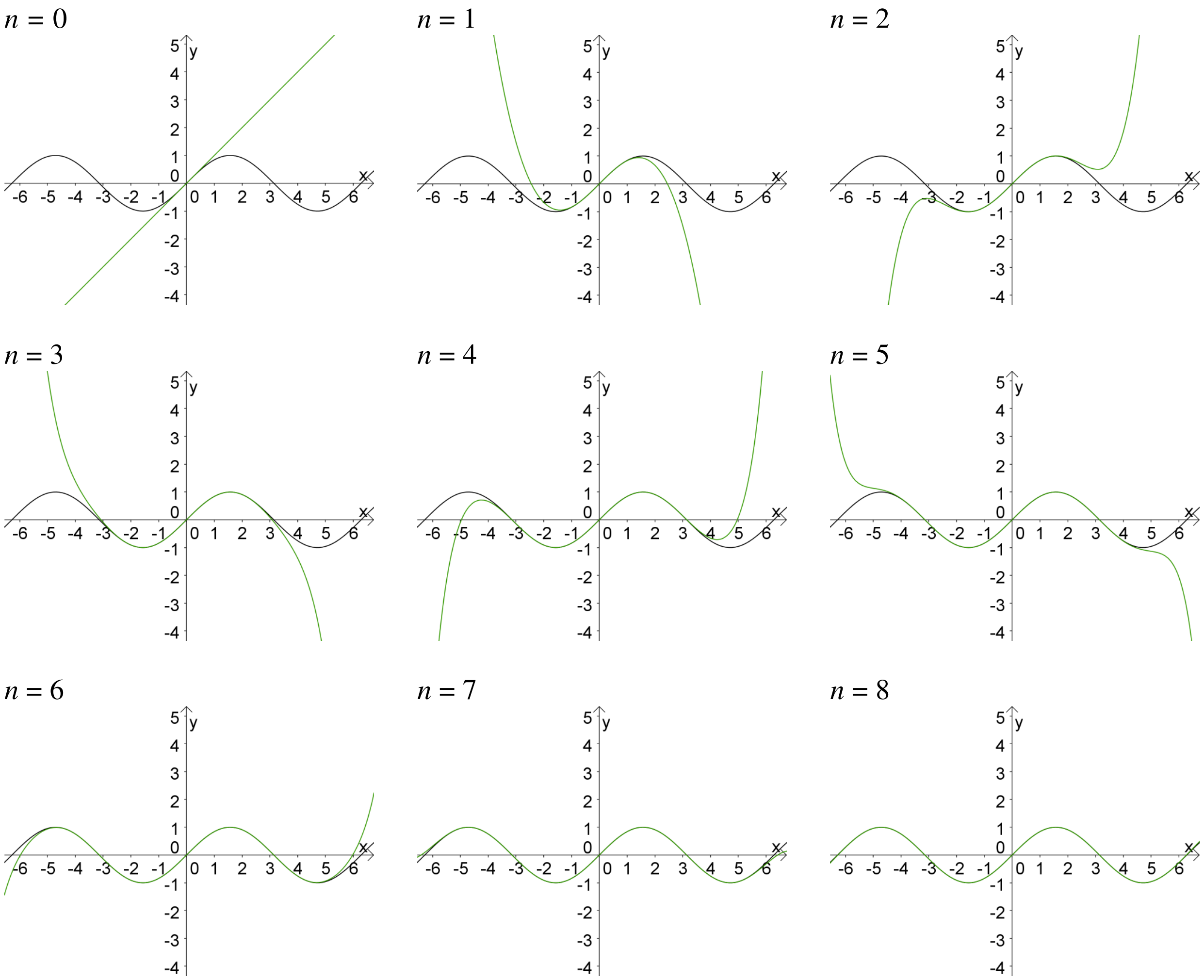


Figure 2.6: Various Maclaurin polynomials for the sine function.

Time for some housekeeping. The polynomials that we have been developing and graphing are called **Taylor Polynomials** after English mathematician Brook Taylor (1685-1731). Every Taylor polynomial has a **center** which in the case of our example has been $x = 0$. When a Taylor polynomial is centered at (or expanded about) $x = 0$, we sometimes call it a **Maclaurin Polynomial** after Scottish mathematician Colin Maclaurin (1698-1746). Neither of these men were the first people to study modeling functions with polynomials, but it is their names that we use. Using this vocabulary, we would say that $P_5(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$ is the fifth-degree Taylor polynomial centered at $x = 0$ for the sine function. Alternately, we could call this the fifth-degree Maclaurin polynomial for the sine function. In this chapter, we will use P (for polynomial) to name Taylor and Maclaurin polynomials, and the subscript will indicate the order of the polynomial. (For now we will use the terms degree and order as synonyms, as you probably did in other math courses. We will say more about the slight difference between these terms in Section 3. In most cases they are the same thing.)

We care about Taylor polynomials chiefly because they allow us to approximate functions that would otherwise be difficult to estimate with much accuracy. If, for example, we want to know the value of sin(2), a unit circle-based approach to the sine function will not be terribly helpful. However, using, for example, the fifth-degree Maclaurin polynomial for the sine function, we see that $\sin(2) \approx 2 - \frac{2^3}{3!} + \frac{2^5}{5!}$, or about 0.933. The calculator value for sin(2) is 0.909297…. Too much error? Use a higher-degree Taylor polynomial. $P_9(2) = 2 - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \frac{2^9}{9!} \approx 0.90935$. However, if we had wanted to approximate $\sin(0.5)$, the fifth-degree polynomial gives $P_5(0.5) = 0.5 - \frac{0.5^3}{3!} + \frac{0.5^5}{5!} = 0.479427$, differing from the calculator's value only in the sixth decimal place. We summarize these observations, largely inspired by Figure 2.6, as follows:

Observation: Taylor Polynomials...

1. … match the function being modeled perfectly at the center of the polynomial.
2. … lose accuracy as we move away from the center.
3. … gain accuracy as we add more terms.

Numbers 1 and 2 in the list above should look familiar to you. They say that Taylor polynomials work a lot like linearization functions. In fact, linearization functions are a special case of Taylor polynomials: first-degree polynomials. You might think there should be a fourth comment in the observation. It seems that as we add more terms, the interval on which the Taylor polynomial is a good approximation increases, gradually expanding without bound. This is certainly what appears to be happening with the sine function. We will have to revisit this idea.

New Polynomials from Old

Suppose we also want a Maclaurin polynomial for the cosine function. We could start over from scratch. That would involve starting with the linearization function $y = 1$ and adding terms, one at a time, hoping to hit upon something that looks good. But this seems like a lot of work. After all, the cosine function is the *derivative* of the sine function. Maybe we can just differentiate the Maclaurin polynomials for sine. Let's see if it works.

$$\sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120}$$

$$\cos x \overset{?}{\approx} 1 - \frac{3x^2}{6} + \frac{5x^4}{120} = 1 - \frac{x^2}{2} + \frac{x^4}{24} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

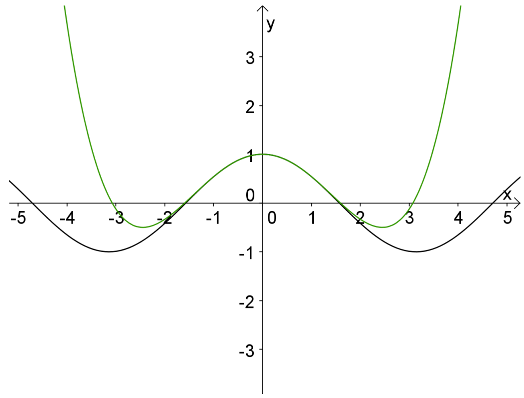


Figure 2.7: $y = \cos(x)$ **and** $y = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

It appears from Figure 2.7 that indeed this polynomial is a good model for the cosine function, at least in the interval $-1.5 < x < 1.5$. If we differentiate all the Maclaurin polynomials of various degrees for the sine function, we can see some general trends in the cosine Maclaurin polynomials: the terms still alternate in sign, the powers are all even (because the terms are derivatives of odd-powered terms), and because of the way that factorials cancel, the denominators are *still* factorials. It appears that

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}. \tag{2.2}$$

Figure 2.8 (next page) shows Maclaurin polynomials for various n. (Again, n is not the degree. In this case, the degree of the polynomial is $2n$.) The table of values that follows the figure also shows numerically how a few of the selected polynomials approximate the cosine function.

The graphs and table demonstrate, once again, that the Maclaurin polynomials match the function being modeled perfectly at the center, that the quality of the approximation decreases as we move from the center, and that the quality of the approximation increases as we add more terms to the polynomial. These big three features of a Taylor polynomial will always be true. We also see that the interval on which the polynomials provide a good approximation of the function being modeled seems to keep growing as we add more terms, just as it did with the sine function.

Here's a random thought. Somewhere (in Algebra 2 or Precalculus) you learned about even and odd functions. Even functions have reflectional symmetry across the y-axis and odd functions have rotational symmetry about the origin. Doubtless, you were shown simple power functions like $f(x) = x^4$ and $g(x) = -2x^3$ as examples of even and odd functions, respectively. At this point, the terminology of even and odd probably made some sense; it derived from the parity of the exponent. But then you learned that the sine function is odd and the cosine function is even. Where are there odd numbers in the sine function? Where are there even numbers in the cosine function? Perhaps the Maclaurin polynomials shed some light on this question.

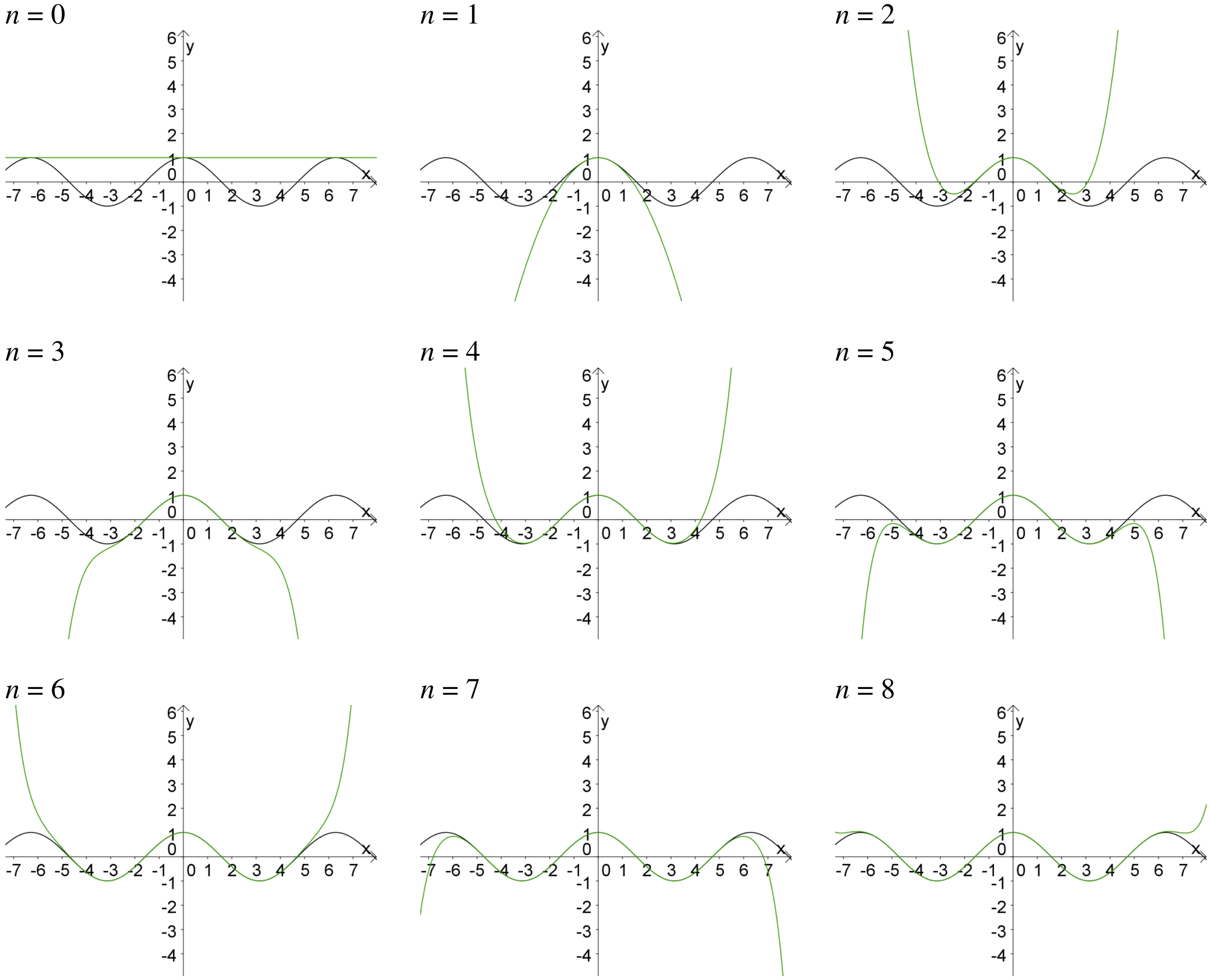


Figure 2.8: Various Maclaurin polynomials for the cosine function

x	0	0.3	0.6	0.9	1.2	1.5	1.8
$P_2(x) = 1 - \frac{x^2}{2}$	1	0.955	0.82	0.595	0.28	-0.125	-0.62
$P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$	1	0.9553	0.8254	0.6223	0.3664	0.0859	-0.1826
$P_8(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$	1	0.9553	0.8253	0.6216	0.3624	0.0708	-0.2271
$f(x) = \cos(x)$	1	0.9553	0.8253	0.6216	0.3624	0.0707	-0.2272

Differentiation isn't the only way that we can turn the Taylor polynomial from one function into the Taylor polynomial for another. We can also substitute or do simple algebraic operations.

Example 1

Find the eighth-degree Maclaurin polynomial for $f(x) = \cos\left(x^2\right)$.

Solution

$f(x)$ is a composition of the cosine function with $y = x^2$. So let us just compose the Maclaurin polynomial for the cosine function with $y = x^2$.

$$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\cos\left(x^2\right) \approx 1 - \frac{\left(x^2\right)^2}{2!} + \frac{\left(x^2\right)^4}{4!} - \frac{\left(x^2\right)^6}{6!} = 1 - \frac{x^4}{2} + \frac{x^8}{24} - \frac{x^{12}}{720}$$

We are asked for only the eighth-degree polynomial, so we simply drop the last term.

$$P_8(x) = 1 - \frac{x^4}{2} + \frac{x^8}{24}$$

A graph of $f(x)$ and $P_8(x)$ is shown in Figure 2.9. ◊

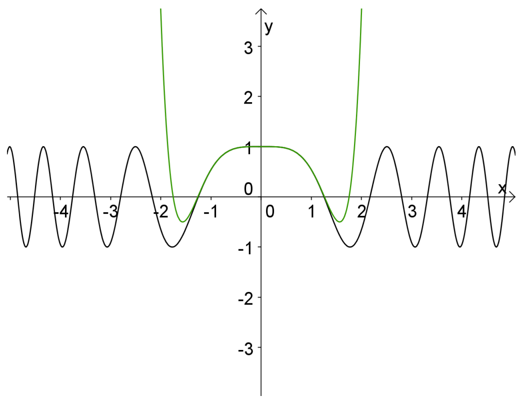


Figure 2.9: $f(x) = \cos\left(x^2\right)$ and $P_8(x)$

Example 2

Use Taylor polynomials to evaluate $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$.

Solution

This question takes us a little ahead of ourselves, but that's okay. This whole section is about laying the groundwork for what is to come. In any event, the first step is to model $f(x) = \dfrac{\sin x}{x}$ with a polynomial. We can do that by starting with a sine polynomial and just dividing through by *x*.

$$\sin x \approx x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5$$

$$\frac{\sin x}{x} \approx \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5}{x} = 1 - \tfrac{1}{6}x^2 + \tfrac{1}{120}x^4$$

Now we claim that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = \lim\limits_{x\to 0}\left(1 - \tfrac{1}{6}x^2 + \tfrac{1}{120}x^4\right)$. This limit is a cinch to evaluate; it's just 1. And this agrees with what you learned at the beginning of your calculus career to be true. ◊

Actually, Example 2 takes us *really* far ahead. The important idea—that we can use Taylor polynomials to simplify the computation of limits—is very powerful. However, there are a lot of ideas being glossed over. Is it really true that the limit of the function will be the same as the limit of its Taylor polynomial? We will dodge this question in Section 10 by looking at Taylor *series* instead of Taylor polynomials. And what about the removable discontinuity at $x = 0$ in Example 2? This is actually a pretty serious issue since $x = 0$ is the *center* of this particular Taylor polynomial. We will explore some of these technical details later, but for now let's agree to look the other way and be impressed by how the Taylor polynomial made the limit simpler.

Some Surprising Maclaurin Polynomials

I would now like to change gears and come up with a Maclaurin polynomial for $f(x)=\dfrac{1}{1-x}$. This actually seems like a silly thing to do. We want Taylor polynomials because they give us a way approximate functions that we cannot evaluate directly. But *f* is a simple algebraic function; it consists of one subtraction and one division. We *can* evaluate $f(x)$ for any *x*-value we choose (other than $x=1$) without much difficulty, so a Taylor polynomial for this function seems unnecessary. We will see, though, that such a polynomial will be very useful.

There are several ways to proceed. We could do the same thing we did with the sine function, though that involves a fair amount of labor and a lot of guessing. We could actually do the long division suggested by the function. That is, we can divide $1\div(1-x)$ using polynomial long division.

$$
\begin{array}{r@{}l}
 & \underline{\;1+x+x^2+\cdots} \\
1-x \big) & 1+0x+0x^2+\cdots \\
 & \underline{1-x} \\
 & \quad x+0x^2 \\
 & \quad \underline{x-\;\;x^2} \\
 & \qquad\quad x^2
\end{array}
$$

This works and is pretty quick, but it is kind of a gimmick since it only applies to a small class of functions. You can play with long division more in the problems at the end of this section.

My preferred approach is to view $\dfrac{1}{1-x}$ as having the form $\dfrac{a}{1-r}$ where a = 1 and r = x. That means, that $\dfrac{1}{1-x}$ represents the sum of a geometric series with initial term 1 and common ratio *x*.

$$\frac{1}{1-x}=1+x+x^2+x^3+x^4+\cdots \tag{2.3}$$

So to obtain a Maclaurin polynomial, we just have to lop off the series at some point. For example $P_3(x)=1+x+x^2+x^3$, $P_4(x)=1+x+x^2+x^3+x^4$, and so forth. Graphs of several Maclaurin polynomials for $f(x)=\dfrac{1}{1-x}$ are shown in Figure 2.10 (next page). This time, *n* really does represent the degree of the polynomial approximator.

As you look at the graphs, a few things should strike you. They are nothing new. All the polynomials appear to match the parent function $f(x)$ perfectly at the center. As you move away from the center, the quality of the approximation decreases, as seen by the green graphs of the polynomials splitting away from the black graph of *f*. Finally, adding more terms seems to improve the quality of the approximation; the higher-order polynomials "hug" the graph of *f* more tightly. The big three features continue to hold. However, we do *not* appear to be able to extend the interval indefinitely as we did with the earlier examples. Even if you graph the 100$^{\text{th}}$-degree Maclaurin polynomial, you will not see a significantly wider interval on which it matches the graph of *f*. It will take us some time to figure out what is going on here. Let's delay this issue until a later section.

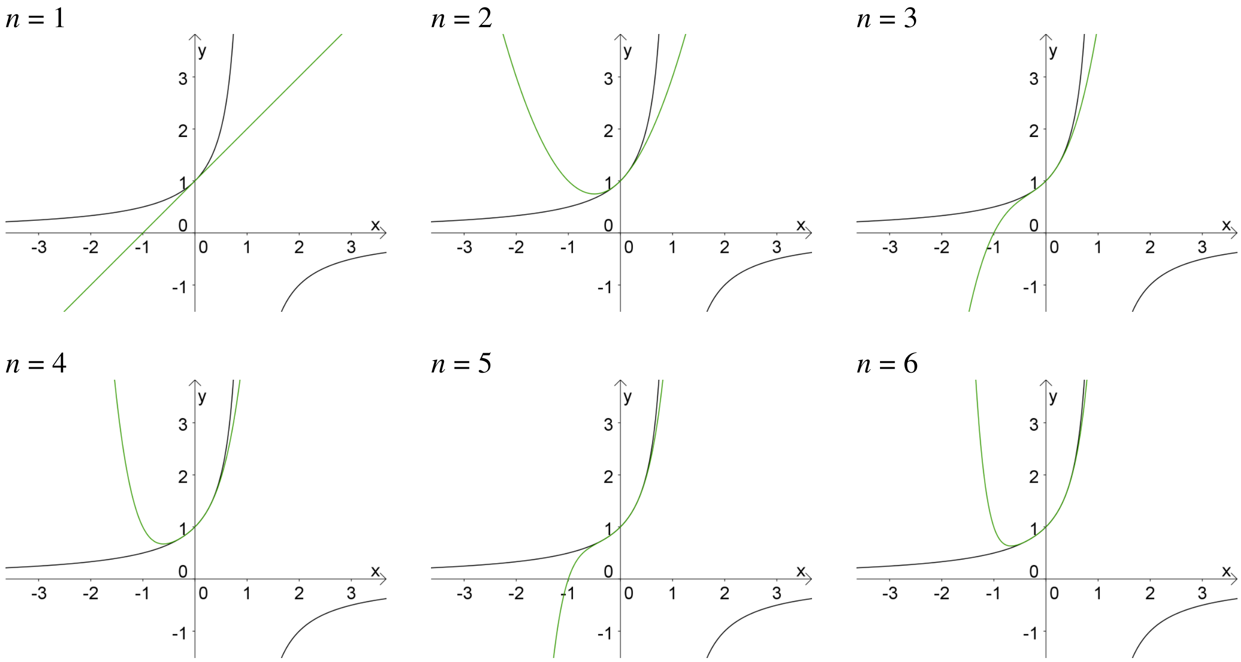


Figure 2.10: Various Maclaurin polynomials for $f(x) = \dfrac{1}{1-x}$

Practice 1

Find third-order Maclaurin polynomials for the following functions:

a. $f(x) = \dfrac{3}{1-2x}$ b. $g(x) = \dfrac{x}{1+x}$ c. $h(x) = \dfrac{4}{2-x}$

Now we must turn to the question of why a Taylor polynomial for $f(x) = \dfrac{1}{1-x}$ is interesting. Well, it isn't. But we can integrate it, and that *will* be interesting. If we know, for example, that

$$\frac{1}{1-t} \approx 1 + t + t^2 + t^3,$$

then it should follow that

$$\int_0^x \frac{1}{1-t}\,dt \approx \int_0^x \left(1 + t + t^2 + t^3\right) dt .$$

t is just a dummy variable here. Carrying out the integration gives the following.

$$-\ln(1-t)\Big|_0^x \approx \left(t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} \right)\Bigg|_0^x$$

$$-\ln(1-x) \approx x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4$$

$$\ln(1-x) \approx -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4$$

From here he we generalize.

$$\ln(1-x) \approx -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \cdots - \tfrac{1}{n}x^n \tag{2.4}$$

And now we have Maclaurin polynomials for a natural logarithm function. (I cheated a bit, ignoring some domain issues with my antiderivative. But if we assume that $x < 1$, we're safe. It turns out, as we will see later, that it makes a lot of sense to restrict our consideration to x-values smaller than 1. So let's just run with this.) If you want to know the value of, say, natural logarithm of 0.3, you can approximate it with a Maclaurin polynomial. Let's use the third-degree Maclaurin polynomial. Note, though, that the polynomial we have is not for ln(x); it's for $\ln(1-x)$. If we want to evaluate ln(0.3), we have to plug in 0.7 for x. Observe:

$$\ln(0.3)=\ln(1-0.7)\approx -0.7-\tfrac{1}{2}(0.7)^2-\tfrac{1}{3}(0.7)^3=-1.0593$$

My calculator says the value of ln(0.3) is about -1.2. So I suppose we should have used a higher-order Taylor polynomial. For you to explore: How many terms do you need to include in the polynomial before the polynomial approximation agrees with the calculator value to two decimal places?

Summary… and a look ahead

In this section, we saw several important ideas, so let us summarize them.

- Many functions can be approximated by polynomials. Depending on the complexity of the original function, this polynomial approximators may be significantly easier to evaluate, and this gives us a way of approximating the values of even the thorniest functions.
- In particular, we found general formulas for Maclaurin polynomials of arbitrary degree for $f(x)=\sin(x)$, $f(x)=\cos(x)$, $f(x)=\dfrac{1}{1-x}$, and $f(x)=\ln(1-x)$.
- We have identified three major characteristics of these polynomial approximators:
    - They are perfect at the center.
    - Their quality decreases as you move away from the center.
    - Their quality improves as you include more terms.
- Finally, we saw that there are several ways to generate new Taylor polynomials from known ones.
    - We can differentiate or integrate, term by term (as we did to find the cosine and logarithm polynomials).
    - We can substitute more complicated expressions for x (as we did to find the polynomial for $f(x)=\cos\left(x^2\right)$).
    - We can manipulate the polynomials algebraically (as we did to find the polynomial for $f(x)=\dfrac{\sin x}{x}$).

Not bad for an overview section.

However, I also feel like we have raised more questions than we have answered. Here are a few questions that I think are natural to be thinking about at this point.

1. *Is there a systematic way to come up with the coefficients of a Taylor polynomial for a given function?* We guessed at the sine function and resorted to some tricks for other functions, but it would be nice to be able to take a given function and a given center and automatically come up with a polynomial that is guaranteed to model the given function well near the center. We will see how to do this in Section 3.
2. *Can we know how big the error will be from using a Taylor polynomial?* As the tables in this section show, Taylor polynomials are not perfect representations for functions; they are only approximations. Whenever we use an approximation or estimation technique, it is important to have a sense for how accurate our approximations are. Section 4 will look into estimating error, and this will be revisited in Section 8.
3. *When can we extend the interval on which the Taylor polynomial is a good fit indefinitely?* It seems like adding more and more terms to the sine and cosine polynomials make better and better

approximations on wider and wider intervals. But for $f(x) = \frac{1}{1-x}$, the interval did not get appreciably bigger after the inclusion of a few terms. Why not? Can we predict which functions can be approximated well for arbitrary x-values and which ones can only be approximated well for certain x-values? This is a tough question, and answering it will occupy most the rest of this chapter.

4. *Can we match a function perfectly if we use infinitely many terms? Would that be meaningful?* This has not come up in the section yet, but you might have wondered whether you could just tack on a "$+\cdots$" the end of equations (2.1), (2.2), or (2.4). Can we just keep adding terms forever? Can we make meaning of that? Section 1 suggests that maybe we can, and in fact this is what we are *really* after in this chapter. It will take us some time to answer this question fully, with most of the work coming in Sections 6, 7, 8, and 10, but I think the payoff will have been worth it.

These four questions frame the rest of the chapter, and we will be returning to them frequently. While there were very few mathematical procedures in this section, there were a lot of big ideas, and the lingering questions are as important as any of the things we think we have learned. In the remaining 8 sections, we will see where answering these questions takes us.

Answers to Practice Problems

1. a. $f(x) = \frac{3}{1-2x}$ is of the form $\frac{a}{1-r}$ with $a = 3$ and $r = 2x$. So we can expand this function as the geometric series $3 + 6x + 12x^2 + 24x^3 + 48x^4 \cdots$. For the third-order Macluarin polynomial, we simply lop off all terms after the cubic. $P_3(x) = 3 + 6x + 12x^2 + 24x^3$

b. $g(x) = \frac{x}{1+x}$ is not quite of the form $\frac{a}{1-r}$ because there is addition in the denominator instead of subtraction. That is easy to fix, though. Rewrite g as $g(x) = \frac{x}{1-(-x)}$. Now the initial term is x and the common ratio is $-x$. Keeping terms up to the third degree, we get $g(x) \approx x - x^2 + x^3$

c. $h(x) = \frac{4}{2-x}$ doesn't have a 1 in the denominator where we would like it. There are two ways to turn that 2 into a 1. The first is to break it up through addition.

$$\begin{aligned} \frac{4}{2-x} &= \frac{4}{1+1-x} \\ &= \frac{4}{1-(x-1)} \end{aligned}$$

This now looks like the sum of a geometric series with $a = 4$ and $r = (x-1)$. Therefore, we suppose that $P_3(x) = 4 + 4(x-1) + 4(x-1)^2 + 4(x-1)^3$. However, this is the wrong answer. To see why, graph $P_3(x)$ in the same window as $h(x)$. You should see that the polynomial is not centered at $x = 0$; it does not match h perfectly there. $P_3(x)$ still does a good job of approximating $h(x)$, just not where we want it to. It *is* a Taylor polynomial, but not a Maclaurin polynomial.

Let's try another strategy to turn the 2 into a 1: dividing.

$$\begin{aligned}\frac{4}{2-x} &= \frac{4/2}{(2-x)/2} \\ &= \frac{2}{1-\frac{x}{2}}\end{aligned}$$

Now it looks like $a = 2$ and $r = x/2$. Indeed, $P_3(x) = 2 + x + \frac{1}{2}x^2 + \frac{1}{4}x^3$ does approximate $h(x)$ near $x = 0$; this is the third-order Maclaurin polynomial for $h(x)$.

Section 2 Problems

1. Find a fifth-degree polynomial that approximates the function $f(x) = e^x$ near the origin. (There are many reasonable answers to this question. You don't need to worry about getting the "right" Maclaurin polynomial just yet, just try to find a fifth-degree function that "looks like" *f* near $x = 0$.)
2. In this section, we found a polynomial for the cosine function by differentiating the polynomial for the sine function. But the opposite of the cosine function is also the *anti*derivative of the sine function. Find a polynomial for the cosine function by integrating the sine function's polynomial and verify that it matches what we obtained in Equation (2.2). Where does the constant term of the cosine polynomial come from?
3. Find the fifth-degree Maclaurin polynomial for $f(x) = \frac{1}{1+x}$ by treating this function like the sum of a geometric series.
4. Find the fifth-degree Maclaurin polynomial for the function $f(x) = \ln(1+x)$. Do this two ways.
   a. Integrate your answer from Problem 3.
   b. Start with the Macluarin polynomial for $\ln(1-x)$ (see Equation (2.4)) and substitute $-x$ for *x*.
   c. Do your answers to (a) and (b) agree?
   d. Graph *f* and your polynomial in the same window. On what interval (roughly) does your polynomial do a good job approximating $f(x)$?
   e. Use your Macluarin polynomial to approximate the value of ln(0.8), ln(1.8) and ln(5). Compare with your calculator's values for these numbers. Which are close? Which are unreasonable?
5. In this problem, you will find and use a Maclaurin polynomial for $f(x) = \arctan(x)$.
   a. Find the sixth-degree Maclaurin polynomial for $f(x) = \frac{1}{1+x^2}$ by substituting into the polynomial for $\frac{1}{1-x}$.
   b. Use your answer to part (a) to find the fifth-degree Maclaurin polynomial for $f(x) = \arctan(x)$.
   c. Graph *f* and your polynomial in the same window. On what interval (roughly) does your polynomial do a good job approximating $f(x)$?
   d. Use your answer to part (b) to approximate the value of arctan(0.2), arctan(-0.6), and arctan(3). Compare with your calculator's values for these numbers. Which are close? Which are unreasonable?
6. In this problem, you will approach trigonometric identities from a new perspective. If you have access to a CAS, that will make the algebra go faster.
   a. Find the fifth-degree Maclaurin polynomial for $\sin(2x)$ by substituting into the polynomial for $\sin(x)$.
   b. Multiply the second-degree Maclaurin polynomial for the $\cos(x)$ and the third-degree Macluarin polynomial for $\sin(x)$, to find a fifth-degree Maclaurin polynomial for $2\sin(x)\cos(x)$. (You'll need to multiply by an extra factor of 2, of course.)
   c. Compare your answers to parts (a) and (b). What do you observe? If you like, use higher-degree Macluarin polynomials to see if you can improve the fit.
   d. Find the sixth-degree Maclaurin polynomial for $f(x) = \sin^2 x$ by squaring the third-degree Maclaurin polynomial for sin(*x*).
   e. Find the eighth-degree Maclaurin polynomial for $f(x) = \cos^2 x$ by

squaring the fourth-degree Maclaurin polynomial for $\cos(x)$.

f. Add your answers to parts (d) and (e). That is, find an eighth-degree Maclaurin polynomial for $f(x)=\sin^2 x+\cos^2 x$. Evaluate your polynomial for x-values near zero. What do you observe? If you like, use higher-degree polynomials to improve the approximation.

7. Find n^{th}-degree Maclaurin polynomials for the following functions. Do this first by treating the functions as sums of a geometric series and second by using polynomial long division. Do the two methods give the same answer?

a. $f(x)=\dfrac{5}{2-x}$, $n=3$

b. $f(x)=\dfrac{3}{1+x^2}$, $n=4$

c. $f(x)=\dfrac{2x}{1+x^2}$, $n=3$

d. $f(x)=\dfrac{x}{2-x}$, $n=3$

8. Find the third-degree Maclaurin polynomial for $f(x)=\dfrac{1}{(1-x)^2}$.

(Hint: $\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right)=\dfrac{1}{(1-x)^2}$.)

9. Find the n^{th}-degree Maclaurin polynomial for the following functions. Then graph the function and the polynomial in the same window.

a. $f(x)=\ln\left(1+x^2\right)$, $n=2$

b. $f(x)=\sin\left(x^3\right)$, $n=3$

10. Shown below (Figure 2.11) are the graphs of $f(x)=e^{-x^2}$ and its second-, fourth-, and sixth-degree Maclaurin polynomials. Determine which graph (A, B, and C) corresponds to which degree Maclaurin polynomial.

11. Shown below (Figure 2.12) are the graphs of $f(x)=\arccos(x)$ and its third- and seventh-degree Maclaurin polynomials. Determine which graph corresponds to which degree Maclaurin polynomial.

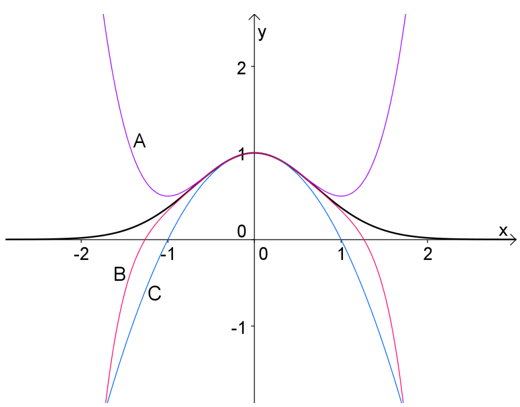


Figure 2.11

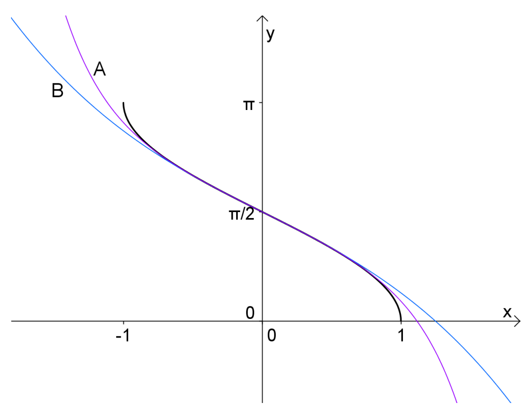


Figure 2.12

Section 3 – A Systematic Approach to Taylor Polynomials

Polynomials

Before getting to Taylor polynomials specifically, we are going to work through several examples just involving polynomials and their derivatives.

Example 1
Find linear functions *f* and *g* such that $f(0)=5$, $f'(0)=-3$, $g(2)=-4$, and $g'(2)=\frac{1}{3}$
Solution
For both these functions, we are given the slope and a point on the line, so it makes sense to use point-slope form, as we almost always do in calculus. For *f* we have $f(x)=5-3x$ and for *g* we have $g(x)=-4+\frac{1}{3}(x-2)$. ◊

You have seen problems like Example 1 before. On the one hand, they are basically initial value problems. You are given *dy/dx* as well as a point (the initial condition), and asked to come up with the function that matches. Procedurally, it is more similar to the tangent line or linearization problem. We are building a line to conform to a particular slope and point. For that matter, this is really more of an algebra problem than a calculus one; you have been doing problems like this for years. Before moving on to our next example, notice how it was useful to keep the $(x-2)$ in $g(x)$ rather than simplifying. Again, this is nothing new, but will be even more helpful now.

Example 2
Find a quadratic function *f* such that $f(0)=3$, $f'(0)=-2$ and $f''(0)=4$.
Also find a quadratic function *g* such that $g(-3)=4$, $g'(-3)=0$, and $g''(-3)=6$.
Solution
Instead of giving the right answer, let's give the most common wrong answer first. Many people generalize the results of Example 1 in a pretty straightforward way.

$$f(x)=3-2x+4x^2$$

Similarly, for *g* the common wrong answer is

$$g(x)=4+0(x+3)+6(x+3)^2.$$

But check it! It should be clear that $f(0)=3$ and $g(-3)=4$, as desired. We also need to compute the derivatives to see if their values match at $x = 0$.

$$\begin{aligned} f(x)&=3-2x+4x^2 & \qquad g(x)&=4+0(x+3)+6(x+3)^2 \\ f'(x)&=-2+8x & g'(x)&=0+12(x+3) \\ f''(x)&=8\neq 4 & g''(x)&=12\neq 6 \end{aligned}$$

While both $f'(0)$ and $g'(-3)$ give the desired values, we are off by a factor of 2 in both cases when it comes to the second derivative. Where did this extra 2 come from? It was the power in the quadratic term that "came down" upon differentiating. To get the correct answer, we need to anticipate that this will happen and divide our quadratic coefficient by two. Thus the correct answers are

$$f(x)=3-2x+\frac{4}{2}x^2 \qquad g(x)=4+0(x+3)+\frac{6}{2}(x+3)^2.$$

I leave it to you to check that the values of the functions and their first derivatives are unaffected by this change and that the values of the second derivatives are now what we want them to be. ◊

Practice 1
Find a quadratic function *f* such that $f(2)=8$, $f'(2)=-1$, and $f''(2)=3$.

Example 3
Find a cubic function *f* such that $f(0)=-1$, $f'(0)=5$, $f''(0)=\frac{1}{5}$, and $f'''(0)=-12$.

Solution
Try it before you read on.

Now that you've tried a bit, I'll give you one hint. The cubic term is <u>not</u> $-4x^3$. If that's what you had, differentiate three times. Your third derivative will not be -12. Try to fix the error before reading further.

The common misconception that leads to $-4x^3$ is simple. In Example 2, we needed to make sure that the quadratic coefficient was half the value of the indicated second derivative. By extension, it would seem that we want the cubic coefficient to be one third the indicated third derivative. That is good inductive reasoning, but it misses the point. The *reason* we had to divide by two was in anticipation of the multiplication by two that was introduced by differentiating a quadratic term. For a cubic term, when we differentiate once, a 3 does "come down," so it is not wrong to divide by three. However, when we differentiate again, the new power, now 2, comes down as well. The correct thing to do, then, is not to divide the -12 by 3, but by 6. (Note that in the third differentiation, the power that comes down is 1, so that does not change what we need to divide by.) The correct answer, then is

$$f(x)=-1+5x+\tfrac{1}{10}x^2-2x^3.$$

Again, I leave it to you to take derivatives and check that they match.

Here is a more formal solution. The function we are seeking has the form $f(x)=c_0+c_1x+c_2x^2+c_3x^3$ where the c_is are unknown numbers. If we take successive derivatives and plug in 0 (the point at which we have information for these functions) we obtain the following.

$$\begin{array}{llll} f(x)= & c_0+c_1x+c_2x^2+c_3x^3 & & f(0)=c_0 \\ f'(x)= & c_1+2c_2x+3c_3x^2 & & f'(0)=c_1 \\ f''(x)= & 2\cdot 1c_2+3\cdot 2c_3x & & f''(0)=2!c_2 \\ f'''(x)= & 3\cdot 2\cdot 1c_3 & & f'''(0)=3!c_3 \end{array}$$

Using the given information, the second column translates to:

$$c_0=-1,\quad c_1=5,\quad 2!c_2=\tfrac{1}{5},\quad 3!c_3=-12$$

Solving for each c_i, we obtain the same coefficients as above. ◊

Notice how the function and derivatives in Example 3 simplified so marvelously when we plugged in 0. There was nothing special about the number zero in this case. No matter the *x*-value for which we are given the values of *f* and its derivatives, this simplification will always happen… as long as we stick to the form we used in Examples 1 and 2. Example 4 demonstrates this.

Example 4
Find the cubic function that has the following properties: $f(1)=0$, $f'(1)=4$, $f''(1)=3$, $f'''(1)=-1$.

Partial Solution
We modify the tableau from Example 3 only slightly.

$$\begin{array}{llll} f(x)= & c_0+c_1(x-1)+c_2(x-1)^2+c_3(x-1)^3 & & f(1)=c_0 \\ f'(x)= & c_1+2c_2(x-1)+3c_3(x-1)^2 & & f'(1)=c_1 \\ f''(x)= & 2\cdot 1c_2+3\cdot 2c_3(x-1) & & f''(1)=2!c_2 \\ f'''(x)= & 3\cdot 2\cdot 1c_3 & & f'''(1)=3!c_3 \end{array}$$

Notice how the second column is essentially the same as in Example 3, just with a 1 plugged in. The critical part, the right-hand side of this column, is identical. The reason is that when we plug $x = 1$ into the k^{th} derivative, all the terms vanish except for the one corresponding to the $(x-1)^k$ term of $f(x)$. ◊

Practice 2
Finish Example 4.

Another noteworthy feature of the methods used in Examples 3 and 4 is that the factorial numbers come up so naturally. Recall that the factorials were major players in some of the polynomials we saw in Section 2. Here we see *why* they came up then; they are the product (literally!) of subsequent differentiation as the powers of the polynomial terms become coefficients. This suggests that there may have been some *reason* to use factorial-related coefficients in the Taylor polynomials of Section 2.

If you can successfully do Practice 3, you will be all set for computing Taylor polynomials.

Practice 3
Find a quartic polynomial *f* that has the following properties: $f(-1)=3$, $f'(-1)=0$, $f''(-1)=-1$, $f'''(-1)=8$, and $f^{(4)}(-1)=10$.
Bonus: Does *f* have a relative extremum at $x = -1$? If so, what kind?

In previous courses, you were probably asked to solve similar questions. Instead of being told the values of a function's derivatives at a point, you were probably given several distinct points on the graph of the function. For example, you may have been asked to find an equation of the parabola that passes through three given points. Or if your teachers were really into having you solve systems of equations, you might even have been asked to find an equation of the cubic polynomial that passes through four given points. The conclusion that you were supposed to draw at the time was that *an* n^{th}*-degree polynomial is uniquely determined by* $n+1$ *points*. In other words, given $n + 1$ points, you can always (excepting degenerate cases) find one and only one n^{th}-degree polynomial whose graph passes through those points. This conclusion makes a certain amount of sense geometrically, but it actually misses a bigger point.

The example and practice problems you have seen so far in this section suggest a broader idea which I suppose is important enough to put in a box.

Observation: To uniquely determine an n^{th}-degree polynomial, we need $n+1$ *pieces of information* about the function.

Being selective about which pieces of information we want to use will give us Taylor polynomials in a moment, but there is good math to think about before we get there. In your previous math classes, the $n+1$ pieces of information were usually distinct points. Sometimes they included the multiplicity of a zero, with each repetition counting as another piece. In the examples we have seen so far, the pieces of information were values of the function and its derivatives all at a particular *x*-value. They could have been a mixture of the two. For example, in a previous course you may have been asked to find an equation of the parabola with vertex (2, 3) that passes through the point (1, 7). At first, it seems like you only have two pieces of information; that's only enough to nail down a first-degree (i.e., linear) polynomial. But the fact that (2, 3) is the *vertex* of the polynomial is also informative; that is the third piece of information. In terms of the previous examples from this section, we are told that the derivative of the function we seek has value 0 at $x=2$. After completing this course, you might solve this problem differently than the way you did in the past.

Practice 4
Do it! Find an equation of the parabola that has vertex (2,3) and that passes through the point (1, 7).

What fascinates me about problems like Examples 3 and 4 is that all the information comes from looking at a single point on the graph of the polynomial. Instead of getting 4 different points to determine a cubic, we look very carefully at one point. Amazingly, all the information that we need can be found there. By knowing the value of the function and its first 3 derivatives at a single point, we recover enough information to build the entire polynomial—for all x. In general, by knowing the value of an n^{th}-degree polynomial function and its first n derivatives at a *single* point, we are able to determine the value of the polynomial at *every* x-value. This is an idea that will continue to build throughout this chapter.

Taylor Polynomials

Using what we have already seen about polynomials in this section, figuring out how to build *Taylor* polynomials should be no problem. The goal, as you recall from Section 2, is to find a polynomial that is a "good match" for a given function, at least near a particular x-value. But in order to do this, we have to agree on what it means to be a good match. Here is the standard that we use:

Taylor Polynomial Criterion: At the center of an n^{th}-degree Taylor polynomial, the values of the polynomial and its n derivatives should match the values of the function being modeled and its n derivatives.

In other words, if f is the function being modeled by a polynomial, P is the Taylor polynomial, and a is the center of the polynomial, we want $P(a) = f(a)$, $P'(a) = f'(a)$, $P''(a) = f''(a)$, and so on all the way to $P^{(n)}(a) = f^{(n)}(a)$ where n is the degree of the polynomial used to model f. If we use the notation $f^{(0)}(x)$ for the zero$^{\text{th}}$ derivative of f, that is for f itself, then the above criteria becomes easy to state.

Taylor Polynomial Criterion (symbolic form): The n^{th}-degree Taylor polynomial for f, centered at a, satisfies $P^{(k)}(a) = f^{(k)}(a)$ for all k from 0 to n.

Let me spend a little time convincing you that this is a good criterion to use. First, it states that we must have $P(a) = f(a)$. This merely says that the polynomial matches the function exactly at its center. This was the first of our "big 3" observations about Taylor polynomials in the previous section. Second, we must have that $P'(a) = f'(a)$, or that the polynomial should have the same slope as the function at the center. If we stop here, this means that the polynomial will be tangent to the function, and we already know that the tangent line provides a good estimate for the values of a function. But we also know that a tangent line is limited for modeling non-linear functions; eventually the functions bend away from the tangent. But what if we bend the tangent line as well so that the concavity of this "line" matches the concavity of the function? Wouldn't that lead to a better model? This is nothing more than requiring that $P''(a) = f''(a)$; the two functions should have the same concavity at a. When we get to the third derivative, it is hard to give a direct graphical description of what it means for $P'''(a) = f'''(a)$, but surely this requirement is not a bad thing. At the very least it says that P's concavity is changing at the same rate as f's, which ought to make P a good model for f. And so we go on, down the line of derivatives, requiring them all to match.

Example 5

Find the third-degree Maclaurin polynomial for $f(x) = e^x$.

Solution

We want to build a third-degree polynomial (so we will need four pieces of information) centered at $x = 0$. We want the derivatives of the polynomial to match the derivatives of f, so we should compute those derivatives at the center.

$$\begin{aligned} f(x) &= e^x & \rightarrow \quad f(0) &= 1 \\ f'(x) &= e^x & \rightarrow \quad f'(0) &= 1 \\ f''(x) &= e^x & \rightarrow \quad f''(0) &= 1 \\ f'''(x) &= e^x & \rightarrow \quad f'''(0) &= 1 \end{aligned}$$

Therefore we want

$$\begin{aligned} P_3(0) &= 1 \\ P_3'(0) &= 1 \\ P_3''(0) &= 1 \\ P_3'''(0) &= 1. \end{aligned}$$

Now this is a problem exactly like all the examples from the first part of this section. If the coefficients of the polynomial are $c_0, c_1, c_2,$ and c_3, then we know we want

$$\begin{aligned} P(0) &= c_0 \\ P'(0) &= c_1 \\ P''(0) &= 2!\cdot c_2 \\ P'''(0) &= 3!\cdot c_3, \end{aligned}$$

or in other words

$$\begin{aligned} P(0) &= c_0 \\ P'(0) &= c_1 \\ \frac{P''(0)}{2!} &= c_2 \\ \frac{P'''(0)}{3!} &= c_3. \end{aligned}$$

But we want P and all its derivatives to have value 1 at $x = 0$, so this means we want $c_0 = 1$, $c_1 = 1$, $c_2 = \frac{1}{2!}$, and $c_3 = \frac{1}{3!}$. Our polynomial approximation is

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

We have answered the question, and I don't want to belabor the issue, but the tableau on the next page summarizes and condenses all the steps that led from the derivatives of f (first column) to the coefficients for use in the polynomial (last column). This setup is a bit of a shortcut, so make sure that you understand how we worked through this example and why the setup provided here is in fact a faithful representation of the work we have done.

$$\begin{array}{ccccc}
f(x)=e^x & \xrightarrow[x=0]{\text{Plug in}} & f(0)=1 & \xrightarrow[\text{factorial}]{\text{Divide by the appropriate}} & \dfrac{1}{0!} \\
f'(x)=e^x & \to & f'(0)=1 & \to & \dfrac{1}{1!} \\
f''(x)=e^x & \to & f''(0)=1 & \to & \dfrac{1}{2!} \\
f'''(x)=e^x & \to & f'''(0)=1 & \to & \dfrac{1}{3!}
\end{array}$$

The function $f(x)=e^x$ and this third-degree Maclaurin polynomial are graphed in Figure 3.1. In addition, a table of values shows the numerical quality of the fit. ◊

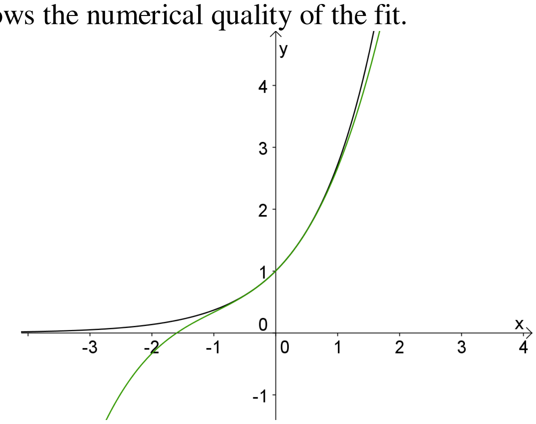


Figure 3.1: $f(x)=e^x$ and $P_3(x)$

x	-0.9	-0.6	-0.3	0	0.3	0.6	0.9
$P_3(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}$	0.3835	0.544	0.7405	1	1.3495	1.816	2.4265
$f(x)=e^x$	0.4066	0.5488	0.7408	1	1.3499	1.8221	2.4596

We can summarize the results of Example 5 in the following definition.

Definition If $f(x)$ is differentiable n times at $x = a$, then its **n^{th}-degree Taylor polynomial centered at a** is given by $P_n(x)=\dfrac{f(a)}{0!}+\dfrac{f'(a)}{1!}(x-a)+\dfrac{f''(a)}{2!}(x-a)^2+\cdots+\dfrac{f^{(n)}(a)}{n!}(x-a)^n$ or $P_n(x)=\displaystyle\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$. If $a = 0$, then we may call the polynomial a Maclaurin polynomial.

And this answers the first of the questions that concluded Section 2. The definition above gives us a systematic method—a formula—for determining the coefficients of a Taylor polynomial. If we follow this formula, that the coefficient of the k^{th}-degree term should be given by $f^{(k)}(a)/k!$, then we are guaranteed (except in particularly nasty situations to be discussed briefly in Section 10) to get a polynomial that is a good model for our function.

Furthermore, this definition allows us to look at Taylor polynomials much more generally than we did in Section 2. All the Taylor polynomials in that section were *Maclaurin* polynomials; we did not yet know enough to stray from $x = 0$ for our center. But now we can find a Taylor polynomial centered anywhere!

Example 6

Find the third-degree Taylor polynomial centered at $x = \frac{2\pi}{3}$ for $f(x) = \sin x$.

Solution

We do not always have to start from the definition to obtain a Taylor polynomial. All of the tricks mentioned in Section 2 (term-by-term differentiation or integration, substitution, algebraic manipulation, using geometric series) still work. However, when we change the center of our polynomial, as we are doing here, we often have to start from scratch. I will do so using the tableau from the end of Example 5.

$$\begin{array}{ccccc}
f(x) = \sin x & \xrightarrow[x=\frac{2\pi}{3}]{\text{Plug in}} & f\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} & \xrightarrow{\text{Divide by the appropriate factorial}} & \dfrac{\sqrt{3}}{2} \\
f'(x) = \cos x & \to & f'\left(\frac{2\pi}{3}\right) = \frac{-1}{2} & \to & \dfrac{-1}{2} \\
f''(x) = -\sin x & \to & f''\left(\frac{2\pi}{3}\right) = \frac{-\sqrt{3}}{2} & \to & \dfrac{-\sqrt{3}}{4} \\
f'''(x) = -\cos x & \to & f'''\left(\frac{2\pi}{3}\right) = \frac{1}{2} & \to & \dfrac{1}{12}
\end{array}$$

Reading off the coefficients we arrive at the polynomial.

$$P_3(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{2\pi}{3}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{2\pi}{3}\right)^2 + \frac{1}{12}\left(x - \frac{2\pi}{3}\right)^3$$

By now you should expect a graph, so here it is (Figure 3.2). As you can see, the graph is a good model for the sine function near $x = \frac{2\pi}{3}$, and the quality of the fit decreases as you move away from the center. I have not included a table for this one because this polynomial is not as useful for approximating values of the sine function. To use this formula, one needs decimal approximations for π and $\sqrt{3}$, and the quality of those approximations would affect the quality of the polynomial approximator for the sine function. So while it may not be as immediately useful as the Maclaurin polynomial, it is still pretty cool that we can create this polynomial. ◊

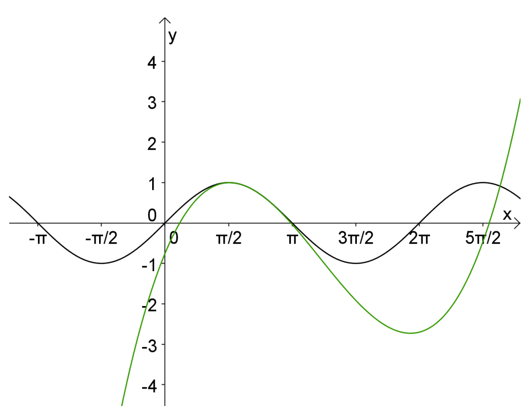


Figure 3.2: $f(x) = \sin(x)$ **and** $P_3(x)$

Practice 5

Find the fifth-degree Taylor polynomial for the cosine function centered at $x = \frac{3\pi}{2}$. Graph both functions in the same window.

At this stage we have a surprisingly large library of "stock" Maclaurin polynomials, so it is a good time to collect them all in one place. Some of them were obtained by "cheating" in Section 2, so you should verify the results below using the definition of a Taylor polynomial. You won't like reading the next sentence. You should memorize the polynomials in this table. At the very least, you should memorize the first four Maclaurin polynomials. They come up quite a bit, and it is good to have a solid handle on them. (The table also indicates where each polynomial was first encountered.)

Before closing this section, I need to make a quick note on the distinction between the **degree** of a Taylor polynomial and the **order** of the Taylor polynomial. For regular polynomials, degree and order are synonyms. For example, $f(x)=x^2+3x-7$ can be described as either a second-degree polynomial or a second-order polynomial. Furthermore, if I tell you a polynomial is sixth-order, it is typically understood that the x^6 term has a non-zero coefficient—there really is such a term in the polynomial—while all terms with higher power have a coefficient of zero so that they vanish. For Taylor polynomials, we have to relax that assumption; an n^{th}-order Taylor polynomial might have a zero coefficient for the x^n term. This comes up when we have Taylor polynomials with missing terms, like in the sine and cosine Maclaurin polynomials. If you were asked to find the second-order Maclaurin polynomial for the sine function, you would find that there was not actually a second-degree term. Let us agree to make the following distinction.

The **degree** of a Taylor polynomial is the highest power in the polynomial. This is exactly in keeping with how you have described degrees of polynomials in previous math classes.

The **order** of a Taylor polynomial is the order of the highest derivative used to compute the polynomial. In many cases, the order will be the same as the degree, but if there are missing terms, the degree may be smaller than the order.

$e^x \approx 1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}$	§3, Example 5
$\sin(x)\approx x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\cdot\frac{x^{2n+1}}{(2n+1)!}$	§2, Equation 1
$\cos(x)\approx 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^n\cdot\frac{x^{2n}}{(2n)!}$	§2, Equation 2
$\frac{1}{1-x}\approx 1+x+x^2+x^3+\cdots+x^n$	§2, Equation 3
$\frac{1}{1+x}\approx 1-x+x^2-x^3+\cdots+(-1)^n x^n$	§2, Problem 2
$\frac{1}{1+x^2}\approx 1-x^2+x^4-\cdots+(-1)^n x^{2n}$	§2, Problem 4
$\ln(1-x)\approx -x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots-\frac{x^n}{n}$	§2, Equation 4
$\ln(1+x)\approx x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n+1}\cdot\frac{x^n}{n}$	§2, Problem 3
$\arctan(x)\approx x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots+(-1)^n\cdot\frac{x^{2n+1}}{2n+1}$	§2, Problem 4

Answers to Practice Problems

1. $f(x) = 8 - (x-2) + \frac{3}{2}(x-2)^2$

2. $f(x) = 0 + 4(x-1) + \frac{3}{2}(x-1)^2 - \frac{1}{6}(x-1)^3$

3. $f(x) = 3 + 0(x+1)\frac{1}{2}(x+1)^2 + \frac{8}{6}(x+1)^3 + \frac{10}{24}(x+1)^4$; the denominator in the final term is 4!.
This function does have an extremum at x = -1. $f'(-1) = 0$, so there is a critical point at x = -1. Furthermore, $f''(-1) = -1 < 0$, so by the second derivative test, f has a local maximum at x = -1.

4. The first way to do this is as you would have in a previous course. We know the vertex, so we will use the "vertex form" of the equation of a parabola: $y = a(x-h)^2 + k$, where (h,k) is the vertex of the parabola. Hence the equation is $y = a(x-2)^2 + 3$. To find the value of a, we plug in the coordinates of the given point for x and y and solve.

$$\begin{aligned} 7 &= a(1-2)^2 + 3 \\ 7 &= a + 3 \\ 4 &= a \end{aligned}$$

So the desired equation is $y = 4(x-2)^2 + 3$.

Another approach is more like what we did in this section. The function we want has the form $y = c_0 + c_1x + c_2x^2$, and $y' = c_1 + 2c_2x$. We know that when x is 1, y is 7, and that when x is 2, y is 3. We also know that when x = 2, y' is 0, because the parabola will have a horizontal tangent at its vertex. These pieces of information give us a system of equations.

$$\begin{aligned} 7 &= c_0 + c_1 + c_2 \\ 3 &= c_0 + 2c_1 + 4c_2 \\ 0 &= \phantom{c_0 +{}} c_1 + 4c_2 \end{aligned}$$

I leave it to you to solve this system to obtain $c_0 = 19$, $c_1 = -16$ and $c_2 = 4$. Hence the desired equation is $y = 19 - 16x + 4x^2$. You can verify that this is equivalent to the answer obtained from the other method.

5. I really like my tableau method, so that's what I'll use.

$$\begin{array}{lcccc}
f(x) = \cos x & \to & f\left(\frac{3\pi}{2}\right) = 0 & \to & \dfrac{0}{0!} \\
f'(x) = -\sin x & \to & f'\left(\frac{3\pi}{2}\right) = 1 & \to & \dfrac{1}{1!} \\
f''(x) = -\cos x & \to & f''\left(\frac{3\pi}{2}\right) = 0 & \to & \dfrac{0}{2!} \\
f'''(x) = \sin x & \to & f'''\left(\frac{3\pi}{2}\right) = -1 & \to & \dfrac{-1}{3!} \\
f^{(4)}(x) = \cos x & \to & f^{(4)}\left(\frac{3\pi}{2}\right) = 0 & \to & \dfrac{0}{4!} \\
f^{(5)}(x) = -\sin x & \to & f^{(5)}\left(\frac{3\pi}{2}\right) = 1 & \to & \dfrac{1}{5!}
\end{array}$$

Therefore we have $P_5(x) = 1\left(x - \frac{3\pi}{2}\right) - \frac{1}{3!}\left(x - \frac{3\pi}{2}\right)^3 + \frac{1}{5!}\left(x - \frac{3\pi}{2}\right)^5$. I leave it to you to graph it.

Section 3 Problems

1. Find the Taylor polynomial of order n centered at a for each of the following functions.

   a. $f(x)=\sqrt{x}$, n = 3, a = 1

   b. $f(x)=e^x$, n = 4, $a = e$

   c. $f(x)=\sqrt{1-x^2}$, n = 2, a = 0

   d. $f(x)=x^3+3x^2-2x+8$, n = 3, a = 0

2. Find the Taylor polynomial of order n centered at a for each of the following functions.

   a. $f(x)=\ln(x)$, n = 4, a = 1

   b. $f(x)=e^{x^2}$, n = 5, a = 0

   c. $f(x)=\sqrt[3]{x}$, n = 2, a = -1

   d. $f(x)=x^2+7x-4$, n = 2, a = 1

3. Find the Taylor polynomial of order n centered at a for each of the following functions.

   a. $f(x)=\dfrac{1}{\sqrt{x}}$, n = 2, a = 4

   b. $f(x)=\cos(2x)$, n = 3, a = 0

   c. $f(x)=\cos(2x)$, n = 3, $a=-\dfrac{\pi}{3}$

   d. $f(x)=\dfrac{1}{x}$, n = 3, a = 5

4. Simplify your answers to Problems 1d and 2d. What do you suppose is true about n^{th}-degree Taylor polynomials for n^{th}-degree polynomial functions?

5. Compare your answer to Problem 2a to the Maclaurin polynomials for $f(x)=\ln(1+x)$ and $f(x)=\ln(1-x)$. What do you notice?

6. Find third-degree Taylor polynomials centered at a for each of the following functions. You may use your answers from previous questions to simplify the work.

   a. $f(x)=x\sqrt{1-x^2}$, a = 0

   b. $f(x)=xe^x$, a = 0

   c. $f(x)=\dfrac{1}{1-x}$, a = 2

   d. $f(x)=\ln(4-x)$, a = 3

7. Find third-order Taylor polynomials centered at a for each of the following functions. You may use your answers from previous questions to simplify the work.

   a. $f(x)=x^3\sin(x)$, a = 0

   b. $f(x)=\dfrac{x}{1+x^2}$, a = 0

   c. $f(x)=\dfrac{x}{1+x^2}$, a = -2

   d. $f(x)=\tan(x)$, a = 0

8. Use a third-degree Taylor polynomial to approximate the value of $\sqrt{10}$. To do this, you will need to (a) pick an appropriate function that can be evaluated to give $\sqrt{10}$, (b) determine an appropriate center for the Taylor expansion of your function, (c) find the Taylor polynomial, and (d) evaluate the polynomial to give an estimate of $\sqrt{10}$. Compare with your calculator's value.

9. Repeat Problem 8, but this time obtain an estimate of $\sqrt[3]{10}$.

10. A function f has the following properties: $f(0)=3$, $f'(0)=-8$, $f''(0)=5$, $f'''(0)=2$. Write the second- and third-order Maclaurin polynomials for f. Use them to approximate $f(0.3)$. Which is most likely a better estimate of the actual value of $f(0.3)$?

11. A function f has the following properties: $f(-4)=2$, $f'(-4)=0$, $f''(-4)=1$, $f'''(-4)=6$. Write the second- and third-order Taylor polynomials centered at x = -4 for f. Use them to approximate $f(-4.2)$. Which is most likely a better estimate of the actual value of $f(-4.2)$?

12. A function *f* has the property that $f^{(n)}(2)=\dfrac{3^n}{(n+1)^2}$ for all non-negative integers *n*. Write the third-degree Taylor polynomial centered at $x = 2$ for *f*.

13. A function *f* has the property that $f^{(n)}(0)=(-1)^n\cdot\dfrac{n^2+1}{n}$ for $n\geq 1$, and $f(0)=6$. Write the second-degree Maclaurin polynomial for *f*.

14. You know only the following information about a particular function *f*: $f(3)=4$, $f'(3)=5$, $f'(2)=8$, $f''(2)=-1$, and $f'''(2)=6$. What is the highest-order Taylor polynomial you can write for this function?

15. The third-degree Taylor polynomial centered at $x = -1$ for a function *f* is given by $P_3(x)=2-(x+1)+(x+1)^2+12(x+1)^3$. Evaluate the following or indicate that there is insufficient information to find the value:

    a. $f(-1)$ c. $f''(0)$

    b. $f'(-1)$ d. $f'''(-1)$

16. The third-degree Taylor polynomial centered at $x = 4$ for a function *f* is given by $P_3(x)=5-4(x-4)+(x-4)^3$. Evaluate the following or indicate that there is insufficient information to find the value:

    a. $f(4)$ c. $f'''(4)$

    b. $f''(4)$ d. $f''(0)$

17. The function *f* has derivatives of all orders. Furthermore, $f(-3)=8$, $f'(-3)=1$, and the graph of *f* has an inflection point at $x=-3$. Write the second-order Taylor polynomial centered at $x=-3$ for *f*.

18. The graph of an infinitely differentiable function *g* has a local minimum at $x = 0$. Which of the following could be the second-order Maclaurin polynomial for *g*? Explain how you know. (Note: options continue to the next column.)

    a. $P_2(x)=5+x+x^2$ c. $P_2(x)=5+x^2$

    b. $P_2(x)=5+x-x^2$ d. $P_2(x)=5-x^2$

19. Find the fourth-degree Taylor polynomials for $f(x)=|x|$ at both $x=2$ and $x=-3$. Why is it not possible to find a Taylor polynomial for *f* at $x = 0$?

20. Figure 3.3 shows the graph of the function *f*. Which of the following could be the second--order Maclaurin polynomial for *f* centered at $x=2$?

    a. $P_2(x)=2+(x-2)-(x-2)^2$

    b. $P_2(x)=(x-2)-(x-2)^2$

    c. $P_2(x)=(x-2)+(x-2)^2$

    d. $P_2(x)=2-(x-2)^2$

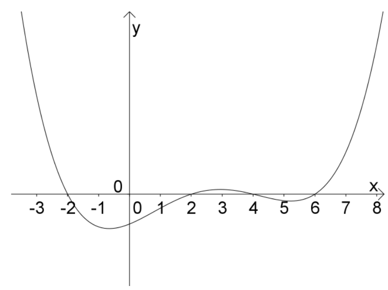


Figure 3.3

21. Let $f(x)=(1+x)^k$ where *k* is a constant. Find the second-order Maclaurin polynomial for *f*.

22. Use your answer to Problem 21 to find second-order Maclaurin polynomials for the following functions.

    a. $f(x)=\dfrac{1}{(1+x)^3}$

    b. $f(x)=\sqrt[5]{(1+x)^2}$

    c. $f(x)=\dfrac{1}{\sqrt{1-x^2}}$

    d. $f(x)=\arcsin(x)$; This time find the third-order polynomial. (Hint: Use your answer to part c.)

Problems 23-26 are True/False. Identify whether the statement is true or false and give reasons and/or counterexamples to support your answer.

23. The n^{th}-order Taylor polynomial for a function can be used to determine the values of the function and its first n derivatives at the center of the polynomial.
24. The n^{th}-order Taylor polynomial for a function can be used to determine the values of the function and its first n derivatives for x-values *other than* the center .
25. The coefficient of $(x-a)^k$ in a Taylor polynomial centered at a is $f^{(k)}(a)$.
26. Taylor polynomials respect horizontal translations. For example, since $\sin x \approx x - \frac{x^3}{6}$, it follows that $\sin(x-1) \approx (x-1) - \frac{(x-1)^3}{6}$.
27. The hyperbolic sine is defined by $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$, and the hyperbolic cosine is defined by $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$. These two functions are useful in many applications, particularly in differential equations.
    a. Find sinh(0) and cosh(0).
    b. Find the derivatives of these two hyperbolic functions.
    c. Find the sixth-order Maclaurin polynomials for the two functions. How are these like and unlike the Maclaurin polynomials of the sine and cosine functions?
28. The hyperbolic tangent is defined as $\tanh x = \dfrac{\sinh x}{\cosh x}$. Show that for small x $\tanh x \approx \tan^{-1} x$ by finding the third-degree Taylor polynomials for each of the two functions.
29. If an object is in free fall, then the only force acting on the object is its own weight. By Newton's second law, this means that $mv' = mg$ (where m is the mass of the object, v is its velocity, and g is a constant).
    a. Assuming that the object is released from rest (so that $v(0) = 0$), show that the solution to this differential equation is $v_1(t) = gt$.
    b. One way to incorporate air resistance into the model is as follows: $mv' = mg - kv^2$, where k is a constant of proportionality. Show that the function $v_2(t) = \sqrt{\dfrac{mg}{k}} \cdot \tanh\left(\sqrt{\dfrac{gk}{m}} \cdot t\right)$ satisfies this differential equation.
    c. Find the third-order Maclaurin polynomial for $v_2(t)$ and compare it to $v_1(t)$. Interpret your findings in terms of the physical context.
30. The so-called "small angle approximation" is the claim that $\sin\theta \approx \theta$ for small values of θ. Justify this approximation using the Maclaurin polynomial for the sine function.*
31. Classical mechanics says that the kinetic energy of a moving object is given by $K_C = \frac{1}{2}mv^2$, where m is the mass of the object and v is its velocity. However, Einstein's Theory of Relativity says that kinetic energy is given by the more complicated formula

$$K_R = mc^2\left(\frac{1}{\sqrt{1-\gamma^2}} - 1\right) \tag{3.1}$$

where c is the speed of light and γ, the "speed parameter" of the object, is given by v/c.

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* The small angle approximation is used often in physics, for example to show that the period of oscillation of a simple pendulum is independent of the mass of the pendulum. For a simple pendulum that has been displaced by an angle θ from its equilibrium position, the magnitude of the restoring force is $F = mg\sin\theta \approx mg\theta$. The remaining details can be found in most physics texts as well as on the internet.

a. Find a second-order Maclaurin polynomial for K_R as a function of γ.

b. Use your polynomial approximation from part (a) to explain why, for speeds much smaller than c, the relativistic kinetic energy reduces to approximately that predicted by the classical model.

32. Figure 3.4 shows an electric dipole created by two oppositely charged particles: the points labeled $+q$ and $-q$. ("Dipole" simply means "two poles," in this case corresponding to the two charged particles.) The particles are separated by a distance of $2d$, and their charges are of the same magnitude (q) but opposite signs. Point P is located r units away from the midpoint of the dipole, as indicated. It can be shown that at point P the magnitude of the electric field caused by the dipole is given by

$$E = k\left[\frac{1}{(r-d)^2} - \frac{1}{(r+d)^2}\right] \tag{3.2}$$

or, if you prefer, by the equivalent

$$E = \frac{k}{r^2}\left[\frac{1}{\left(1-\frac{d}{r}\right)^2} - \frac{1}{\left(1+\frac{d}{r}\right)^2}\right] \tag{3.3}$$

where in both cases k is a constant that depends on the units chosen.

a. Replace both fractions in Equation (3.3) with second-order Maclaurin approximations, using $\frac{d}{r}$ as the variable.

b. Use your answer to part (a) to show that the magnitude of the electric field at a distance of r units from the center of the dipole is proportional to $1/r^3$. Thus electric fields caused by dipoles follow an inverse-cube law. (This is in contrast to electric fields caused by a single point charge; those follow an inverse-square law.)

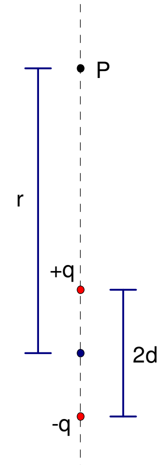

Figure 3.4: An electric dipole

33. The Doppler Effect is the reason why an ambulance or police siren appears to change pitch when it passes you on the road. Though the siren gives off sound at a particular frequency, the relative motion of you and the siren affect how you perceive the sound. The Doppler Effect is described quantitatively by the equation

$$f_{obs} = f_{act} \cdot \frac{343 + v_D}{343 - v_S} \tag{3.4}$$

where f_{obs} is the observed or perceived frequency of the siren, f_{act} is the actual frequency of the siren, v_D is the velocity of the detector (you), v_S is the velocity of the source (or siren), and 343 is the speed of sound in meters per second. Equation (3.4) assumes that the detector and the source are approaching one another. If they are moving away from one another, the signs of v_D and v_S are both flipped.

If both the detector and the source are moving at speeds much less than 343 meters per second, which is likely, Equation (3.4) can be simplified to

$$f_{obs} \approx f_{act} \cdot \left(1 + \frac{v_S + v_D}{343}\right) \tag{3.5}$$

as you will show in this problem.

a. Rewrite Equation (3.4) as

$$f_{obs} = f_{act} \cdot (343 + v_D)\, g(v_S) \tag{3.6}$$

where $g(v_S) = \dfrac{1}{343 - v_S}$. Find a first-degree Maclaurin polynomial for g.

b. In Equation (3.6), replace $g(v_S)$ with your answer from part (a).

c. Use your answer from part (b) to obtain Equation (3.5) as a first-order Taylor polynomial for f_{obs}. (Note that any terms of the form $k \cdot v_D v_S$ are considered second-order because they are the product of two first-order terms.)

Section 4 – Lagrange Remainder

Section 2 closed with a few big questions about Taylor polynomials. Section 3 answered the first of these questions; it showed us how to compute the coefficients for a Taylor polynomial systematically. In this section we turn to the second question. How do we determine the quality of our polynomial approximations?

There is one trivial answer to this question: Get out a calculator and punch stuff in. But this misses some of the bigger ideas. For one thing, one has to wonder how the calculator is coming up with its decimal approximations to begin with. For another, we sometimes do not have an explicit expression for the function we are modeling. This will come up in particular in Section 10, but we will see some examples in the problems in this section. (There were a few in the last section too.) Even in these cases, we may be able to estimate how accurate a Taylor polynomial's approximation is. For these reasons, our approach to analyzing error will be based on the assumption that our calculators are mainly useful for arithmetic. I know this seems silly, but play along and you will learn something along the way.

Taylor's Theorem

First we need to develop a little bit more notation. We already have $P_n(x)$; it is the n^{th}-order Taylor polynomial for a specified function (with a specified center as well). We will also use $R_n(x)$. Here *R* stands for remainder. The remainder term accounts for the disagreement between the actual values of $f(x)$ and the approximated values $P_n(x)$; it is what remains of $f(x)$ after you have computed $P_n(x)$.

Note two things about $R_n(x)$. First, it depends on *n*. As we increase *n* (i.e., add more terms to the polynomial), we expect the remainder to decrease in size. Second, $R_n(x)$ depends on *x*. As we move away from the center of the polynomial, we expect that the size of $R_n(x)$ will usually increase.

In short, for any *x*-value in the domain of *f*, we have $f(x) = P_n(x) + R_n(x)$. All this says is that the actual value of *f* at some *x*-value is equal to our polynomial approximation at *x* plus some remainder.

We will almost never know the function $R_n(x)$ explicitly. Coming up with such a function is just too tall an order. However, that does not mean we cannot know anything about it. One fundamental fact about $R_n(x)$ is given by the following theorem.

Theorem 4.1 – Taylor's Theorem with Lagrange Remainder
If *f* is differentiable $n + 1$ times on some interval containing the center *a*, and if *x* is some number in that interval, then

$$f(x) = P_n(x) + R_n(x). \tag{4.1}$$

Moreover, there is a number *z* between *a* and *x* such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}. \tag{4.2}$$

That is, there is some *z* between *a* and *x* such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}. \qquad (4.3).$$

There are many expressions that give the remainder term $R_n(x)$. The form in Equation (4.2) is called the Lagrange form of the remainder. We will see another form in Section 5.

One thing that is important to realize about Theorem 4.1 is that it is an *existence* theorem. It tells us that there must be some number z with the property expressed in Equation (4.2), but it does not tell us how to find that number. In the vast majority of cases, in fact, we will be completely unable to find the magic z. In other words, Theorem 4.1 is all but useless for computation. This is a bit of a let-down since working with Taylor polynomials is all about computation. The real purpose to which we will direct Theorem 4.1 is to convert it to a practical way of estimating the error involved in a Taylor polynomial.

Theorem 4.1 is a hard theorem, both to prove and to understand. At the end of this section are some of my thoughts as to how we might make meaning of the mysterious z in the Lagrange remainder, but for now we proceed to its most useful consequence.

Lagrange Error Bound

Because of the difficulties in finding the number z in Theorem 4.1, the best we can hope for is an estimate on the remainder term. This may seem like settling, but it actually makes some sense. We use Taylor polynomials to approximate the values of functions that we cannot evaluate directly. We are already settling for an approximation. If we could find the exact error in our approximation, then we would be able to determine the exact value of the function simply by adding it to the approximation. If we were in a situation where we could find the exact value of the function, then why were we starting off with an approximation at all?!

Our first order of business is to convert the existence-based Lagrange remainder to something we can use. This is accomplished by the following theorem.

Theorem 4.2 – Lagrange Error Bound*
Suppose $x > a$. If $f^{(n+1)}(x)$ is bounded on the interval $[a,x]$ (i.e., if there is a positive number M such that $-M \le f^{(n+1)}(t) \le M$ for all t in $[a,x]$), then $\left|R_n(x)\right| \le \frac{M}{(n+1)!}\left|x-a\right|^{n+1}$.
If $x < a$, then the interval in question is simply to be replaced with $[x,a]$. No other change is required.

Proof
The proof for this theorem is both reasonably straightforward and quite instructive, so let's take a look. In the proof, we will assume that $x > a$ to simplify the details, but the theorem holds just as well for $x < a$.

Since
$$-M \le f^{(n+1)}(t) \le M$$
for all t in the interval (note that t is just a dummy variable), we can multiply through by $\frac{(x-a)^{n+1}}{(n+1)!}$ to obtain
$$\frac{-M}{(n+1)!}\cdot(x-a)^{n+1} \le \frac{f^{(n+1)}(t)}{(n+1)!}\cdot(x-a)^{n+1} \le \frac{M}{(n+1)!}\cdot(x-a)^{n+1}. \tag{4.4}$$
The manipulation to obtain Inequality (4.4) is justified since $\frac{(x-a)^{n+1}}{(n+1)!} > 0$. Since Inequality (4.4) holds for all t in the interval, it holds in particular for the value z guaranteed by Theorem 4.1. (This is often how existence theorems like Theorem 4.1 work in mathematics. They are used to prove other theorems.) Therefore we have

* The term "Lagrange Error Bound" is in common use, but is actually a bit misleading in my opinion. The name comes from the fact that the expression in the theorem follows quite naturally from the Lagrange form of the remainder term. However, this error bound can also be derived from other forms of the remainder and even from analyses that assume no explicit form for the remainder, as you will see in Problem 21.

$$\frac{-M}{(n+1)!}\cdot(x-a)^{n+1}\le\frac{f^{(n+1)}(z)}{(n+1)!}\cdot(x-a)^{n+1}\le\frac{M}{(n+1)!}\cdot(x-a)^{n+1}$$

or

$$\frac{-M}{(n+1)!}\cdot(x-a)^{n+1}\le R_n(x)\le\frac{M}{(n+1)!}\cdot(x-a)^{n+1}.$$

Another way to write this compound inequality is to state that $\left|R_n(x)\right|\le\left|\frac{M}{(n+1)!}\cdot(x-a)^{n+1}\right|$. However, the only part of the right side that can possibly be negative is the quantity $(x-a)$. (Even this is not negative under the assumptions of our proof, but if we were to go through the details for $a>x$, then we would need to worry about the sign.) We pull all the other positive stuff out of the absolute value bars and we are left with

$$\left|R_n(x)\right|\le\frac{M}{(n+1)!}\left|x-a\right|^{n+1} \tag{4.5}$$

as desired. ◊

A more conceptual approach to the Lagrange error bound is this. Since we know that $f^{(n+1)}(z)$ is no more than *M* in absolute value, we replace it with *M*, sacrificing equality for inequality. Then we have to throw in some absolute value bars to avoid getting caught up in pernicious signs. (We wouldn't want to accidentally say that a negative number is bigger than a positive number, for example.) In this context, absolute value bars around $R_n(x)$ are actually kind of handy. $\left|R_n(x)\right|$ gives the absolute error as opposed to the signed error between $f(x)$ and $P_n(x)$.

The Lagrange error bound frees us from the need to find *z*, but it replaces *z* with *M*. Many times, *M* is no easier to find than *z*. The difficulty in using the Lagrange error bound is to find a reasonable upper bound—a cap—on the values of $f^{(n+1)}(t)$ on the interval in question.

Example 1

Use the third-order Maclaurin polynomial for $f(x)=\sin(x)$ to approximate sin(0.5). Estimate the error involved in this approximation.

Solution

As we know for this function, $P_3(x)=x-\frac{x^3}{6}$. Therefore $\sin(0.5)\approx 0.5-\frac{(0.5)^3}{6}=\frac{23}{48}$ or 0.47916666…. To estimate the error, we use the Lagrange error bound.

$$\left|R_3(0.5)\right|\le\frac{M}{(3+1)!}(0.5-0)^{3+1}=\frac{M}{384}$$

But what do we use for *M*? Recall that *M* is a bound for the $(n+1)^{\text{st}}$ derivative of *f* on the interval $[a,x]$. In this case, the fourth derivative of *f* is given by $f^{(4)}(x)=\sin(x)$. Thus *M* is a number such that $-M\le\sin(t)\le M$ for *t*-values between 0 and 0.5. In fact, we can choose *any* number *M* that is certain to satisfy this inequality. We know that the sine function is bounded by 1 for *all t*, so we can use 1 for *M*: $\left|R_3(0.5)\right|\le\frac{1}{384}$. The error in our approximation is no more than 1/384 or 0.00260416666….

But don't take my word for it. The calculator concurs! Our approximation for sin(0.5) was about 0.4792. The calculator puts the value at 0.4794. The difference between these two numbers is about 0.0002 which is truly less than 1/384. ◊

As Example 1 shows, the tricky thing about using Lagrange error bound is finding a suitable value for *M*. In fact, we didn't use the "best" possible value of *M* in this example. Since $0.5 < \frac{\pi}{6}$, $\left|f^{(4)}(t)\right| = \left|\sin t\right| < \frac{1}{2}$ for all *t*-values in the interval $[0, 0.5]$. So we could have used 1/2 for *M*. This would have shown us that our approximation for sin(0.5) was in truth more accurate than we initially thought based on using *M* = 1. However, in this problem the *actual* error in our approximation would still have been an order of magnitude less than our estimated error. The point is that it often doesn't pay to spend a lot of time and effort trying to find the smallest *M* that you can. What we really need of *M* is just that it is truly an upper bound for the values of the $(n+1)^{\text{st}}$ derivative and that it is reasonably small—the "reasonable" part will depend on context.

Practice 1
Use the sixth-order Maclaurin polynomial for the cosine function to approximate cos(2). Use the Lagrange error bound to give bounds for the value of cos(2). In other words, fill in the blanks: $\_\_\_\_ \le \cos(2) \le \_\_\_\_$.

Example 2
Use the third-degree Maclaurin polynomial for $f(x) = e^x$ to approximate $\sqrt{e}$. About how much error is there in your approximation? What degree polynomial is guaranteed to give error less than 0.001?
Solution
The third-degree Maclaurin polynomial for $f(x) = e^x$ is $P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$, and $\sqrt{e} = e^{1/2}$. We want to approximate $f\left(\frac{1}{2}\right)$.

$$f\left(\tfrac{1}{2}\right) \approx P_3\left(\tfrac{1}{2}\right) = 1 + \frac{1}{2} + \frac{(1/2)^2}{2!} + \frac{(1/2)^3}{3!} = \frac{79}{48} \text{ or } 1.645833333\ldots$$

For the error, we use the Lagrange error bound: $\left|R_3\left(\frac{1}{2}\right)\right| \le \frac{M}{4!}\left(\frac{1}{2} - 0\right)^4 = \frac{M}{384}$, where *M* is an upper bound for the values of the fourth derivative of $f(x) = e^x$ on the interval in question. $f^{(4)}(x) = e^x$, so we need to find a cap for the values of e^t for *t* in $\left[0, \frac{1}{2}\right]$. We know that $f(t) = e^t$ is an increasing function, so it will take on its largest value at the right endpoint of the interval. Hence a choice for *M* might be $e^{1/2}$. But this is the value we are trying to approximate, so that won't do. We need some reasonably small number for *M* that is bigger than $e^{1/2}$. Well, $e^{1/2}$ is the square root of *e*, so $e^{1/2}$ is clearly less than *e*. In turn, *e* is less than 3. We can use 3 for *M* and call it a day. The error in our approximation is less than 3/384 = 1/128.

(We could do better for *M* if we really wanted to. Since $e < 4$, $\sqrt{e} < \sqrt{4}$. So $e^{1/2} < 2$. Two would be a better value for *M* than 3. It tells us that our error is actually les than 1/192. This is a better answer, but it is also more work and requires a bit more cleverness. If what we care about is an order of magnitude for the error, as we're about to, then the extra work doesn't really pay.)

Our approximation for $\sqrt{e}$ is not guaranteed to be within 0.001 of the actual value; we would need a higher-degree Maclaurin polynomial to be sure of that level of accuracy. We can still figure out how high a degree is needed by looking at the Lagrange error bound. We want the error to be less than 0.001. That means we want

$$\left|R_n\left(\tfrac{1}{2}\right)\right| \le 0.001.$$

But we know that

$$\left|R_n\left(\tfrac{1}{2}\right)\right| \le \frac{M}{(n+1)!}\left(\tfrac{1}{2} - 0\right)^{n+1} = \frac{M}{2^{n+1}(n+1)!}.$$

If we insist that this latter expression be less than 0.001, we will certainly have $\left|R_n\left(\tfrac{1}{2}\right)\right| \le 0.001$ as desired. In keeping with the parenthetical remarks above, we will use $M = 2$ just to simplify the fraction a bit. (If it weren't for this simplifying benefit of $M = 2$, I would have happily stuck with my initial M-value of 3.) Note that it is important in this example that all derivatives of the function $f(x) = e^x$ are the same. If they were not, the value of M might change with n. This would require a more careful problem-solving strategy. In any event, we must solve

$$\frac{1}{2^n (n+1)!} < 0.001 .$$

The factorial function is not easily inverted, so your best bet is just to look at a table of values for the expression on the left-hand side.

n	0	1	2	3	4	5
$\frac{1}{2^n (n+1)!}$	1	0.25	0.04167	0.00521	0.00052	0.00004

As we see, the expression is first less than 0.001 when $n = 4$, so a fourth-degree Maclaurin polynomial is sure to estimate $\sqrt{e}$ to within 0.001. (Note, sometimes this question is a trick question, asking for the *number of terms* needed instead of the *degree* of the polynomial. If that were the case here, then our answer would be 5, not 4. A fourth-degree polynomial with no missing terms has 5 terms.) ◊

Example 3

For a particular function, it is known that $f(2) = 8$, $f'(2) = 5$, and $\left|f''(x)\right| \le 3$ for all x-values in the domain of f. Approximate $f(2.5)$ and estimate the amount of error in the approximation.

Solution

From the given information, $P_1(x) = 8 + 5(x-2)$. $f(2.5) \approx P_1(2.5) = 8 + 5(2.5-2) = 10.5$

The Lagrange error bound in this case is $\left|R_1(2.5)\right| \le \frac{M}{2!}(2.5-2)^2 = \frac{M}{8}$. The choice for M is made simple by the given information. We know that 3 is an upper bound for the second derivative, so we take 3 for M. This means that our error is no more than 3/8. ◊

Example 3 may seem silly or contrived. But consider this situation. Suppose you know a car's position and velocity at a particular point in time. Suppose you also know the maximum acceleration of the car and you want to determine where it will be at some later time. This is exactly the situation that Example 5 would model (given some units, of course). You can imagine how predicting the location of the car with some estimate of error might be useful for, say, a GPS trying to find you after you have entered a tunnel and lost touch with the satellites.

Practice 2

Approximate the value of $\ln(1.5)$ by using a third-order Taylor polynomial for $f(x) = \ln(x)$ centered at $x = 1$. Estimate the amount of error in this approximation.

Thoughts on Taylor's Theorem with Lagrange Remainder

Though working with Lagrange error bound is the most significant idea to take away from this section, I would like to revisit Taylor's Theorem with Lagrange remainder and try to convince you that it makes sense. As I have said, we generally cannot find z, but let's look at an example where we can. The example will have to be ridiculously simple (on a relative scale, of course); the purpose is not necessarily to provide a model for future problem solving, but to develop some understanding of what the Lagrange remainder is about.

Example 4

Let $f(x) = -x^3 + 4x + 1$.

a. Use the 0^{th}-order Maclaurin polynomial for *f* to approximate $f(1)$.

b. Find $R_0(1)$.

c. Find the number z guaranteed by Theorem 4.1.

Solution

a. For the moment, we will ignore the fact that we can compute $f(1)$ directly. Instead, we find $P_0(x)$ which will simply be $f(0)$. $f(0) = 1$, so $P_0(x) = 1$. Our 0^{th}-order approximation of $f(1)$ is simply $P_0(1) = 1$. We estimate that $f(1) \approx 1$.

b. In reality $f(1) = 4$. We were off by 3. $R_0(1) = f(1) - P_0(1) = 4 - 1 = 3$.

c. Finally, we are asked to find a number z such that $\frac{f^{(1)}(z)}{1!} \cdot (1-0)^1 = R_0(1)$. $f^{(1)}(x)$, or $f'(x)$ as it is more commonly known, is given by $f'(x) = -3x^2 + 4$ The equation $\frac{f^{(1)}(z)}{1!} \cdot (1-0) = R_0(1)$ becomes $-3z^2 + 4 = \frac{3}{1-0}$.

$$\begin{aligned} -3z^2 + 4 &= 3 \\ -3z^2 &= -1 \\ z^2 &= \tfrac{1}{3} \\ z &= \sqrt{\tfrac{1}{3}} \end{aligned}$$

We ignore the other solution to this quadratic because it is not between *a* (0) and *x* (1). ◊

If the computations in part (c) of Example 4 seem familiar to you, they should; you've done them before. Finding z in this example was exactly the same as the Mean Value Theorem problems you have worked in the past. Take another look. We are trying to find a number z such that $f'(z)$ is equal to some previously computed value. Moreover, that value is in fact $\frac{f(1)-f(0)}{1-0}$.This is <u>identical</u> to "Find the number guaranteed by the Mean Value Theorem" problems. This means that when $n = 0$, Taylor's theorem as presented in Theorem 4.1 *is* the Mean Value Theorem. In fact, more can be said. The Lagrange form of the remainder can be interpreted as a way of generalizing the MVT to higher-order derivatives. This is a subtle point, so we will work through it slowly.

The Mean Value Thorem says that there exists a z between a and x such that $f'(z) = \frac{f(x) - f(a)}{x - a}$. Put another way, this says that

$$f(x) = f(a) + f'(z)(x - a). \tag{4.6}$$

Let us look at this in light of Example 4 where $a = 0$ and $x = 1$. Figure 4.1 shows a graph of *f* along with its 0^{th}-order Maclaurin polynomial $P_0(x)$. Now clearly $f(1)$ is different from $f(0)$. The difference between them can be accounted for by a linear function. That is, there is a line connecting the points $(0, f(0))$ and $(1, f(1))$. (This is the dashed secant segment in Figure 4.1.) Of course, two points always determine a line. The significance of the MVT is to tell us something important about that line: its slope is a number obtained by $f'(x)$ at some point z in the interval (a, x).

Now let's connect this to the Lagrange remainder. Since our Taylor polynomial is a constant function, it approximates the actual value of *f*(*x*), namely 4, with the value of *f*(*a*), namely 1. Of course, there is error in this approximation. What the Lagrange remainder tells us is that the error can be exactly accounted for by tacking on a linear term to the polynomial approximation. The only catch is that the coefficient of this linear term is not the first derivative of *f* at 0, as it would be if we were computing another term in the Taylor polynomial. This makes sense; if we included the first-order term of the Taylor polynomial, then we would still have only an *approximation* to $f(1)$. However, Theorem 4.1 tells us there is *some* point at which the first derivative of *f* gives the slope we need to match $f(1)$ exactly. And this point is the same as the one guaranteed by the Mean Value Theorem.

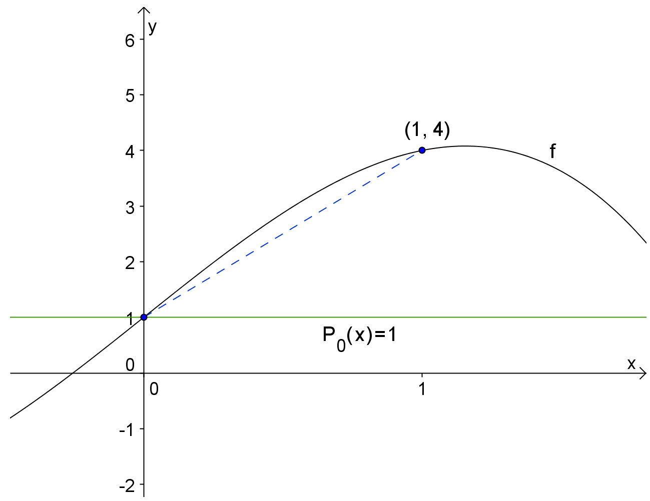

Figure 4.1: $f(x) = -x^3 + 4x + 1$ and $P_0(x)$

The slope of that blue segment in Figure 4.1 is exactly what we need in order to get from the point $(0,1)$ to the point $(1,4)$. The MVT and Theorem 4.1 are two different ways of telling us that $f'(x)$ actually takes on this value at some point in the interval $(0,1)$: at $x = \sqrt{\frac{1}{3}}$ according to Example 1.

Let's try moving up to $n = 1$. Taylor's Theorem for $n = 1$ states that there is a number z between a and x such that

$$\begin{aligned} f(x) &= P_1(x) + R_1(x) \\ &= f(a) + f'(a)(x-a) + \frac{f''(z)}{2!}(x-a)^2. \end{aligned}$$

This means that there is a quadratic term that accounts exactly for the difference between the exact value of $f(x)$ and the linear polynomial approximation $P_1(x)$. Again, though, we do not compute the coefficient of this quadratic term by looking at $f''(a)$; that would just give us the quadratic Taylor approximation. Instead, we are told that there is some other point at which the value of $f''(x)$ is exactly what we need to account for the error.

Example 5

Revisit Example 4, this time using a first-order Maclaurin polynomial to approximate $f(2)$. Find the value of z guaranteed by Theorem 4.1.

Solution

For $f(x) = -x^3 + 4x + 1$, the first-order Maclaurin polynomial is $P_1(x) = 1 + 4x$. $f(2) \approx P_1(2) = 9$. This is actually an even worse approximation than in Example 4 since the actual value for $f(2)$ is 1. (How can this be? How can a higher-order Taylor polynomial lead to a *worse* approximation?) We can now compute the value of the remainder: $R_1(2) = f(2) - P_1(2) = 1 - 9 = -8$. By Taylor's Theorem with Lagrange remainder there exists a number z between 0 and 2 such that $\frac{f''(z)}{2!}(2-0)^2 = -8$. Let's solve, noting that $f''(x) = -6x$.

$$\frac{f''(z)}{2!}\cdot 2^2 = -8$$
$$f''(z) = -4$$
$$-6z = -4$$
$$z = \tfrac{2}{3}$$

Figure 4.2 shows *f* along with its first-order polynomial and, dashed and in blue, the quadratic function whose second-order term was computed using $f''\left(\tfrac{2}{3}\right)$. Notice that the blue curve lands right at $(2,1)$. ◊

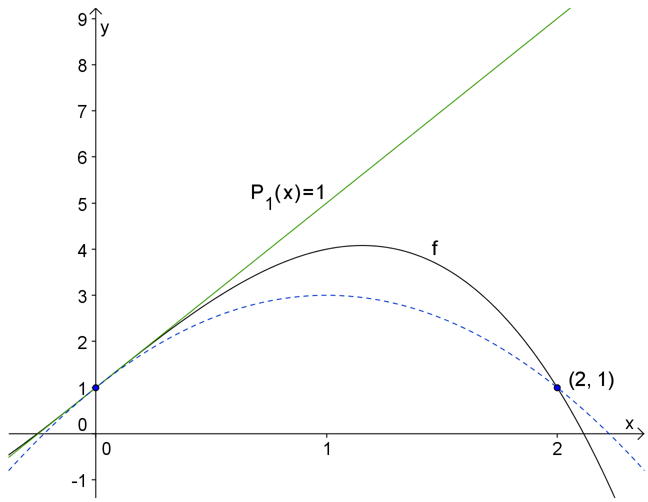

Figure 4.2: $f(x) = -x^3 + 4x + 1$ **and** $P_1(x)$

I want to stress that Examples 4 and 5 are only meant to provide some concreteness to the idea of Lagrange remainder. The actual computations involved are almost irrelevant since we may never be able to carry them out in practice. (Don't believe me? Try to find z in the case where the fourth-order Maclaurin polynomial for $f(x) = e^x$ is use to approximate $e^{2.5}$.) Rather, the point was just to establish the reasonableness of the claim that for an n^{th}-degree Taylor polynomial and a particular *x*-value there is a magic number z that can be used to make a polynomial that exactly computes $f(x)$:

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}. \qquad (4.7)$$

It is the right side of Equation (4.7) that, when graphed, produces the dashed, blue curves that land unfailingly at the point $\left(x, f(x)\right)$. In this way, use of the number z accounts for the error of the Taylor polynomial:

$$\frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1} = f(x) - P_n(x).$$

Summary

Taylor's Theorem (Theorem 4.1) is a pretty remarkable theorem. It tells us that polynomials can be used to approximate all kinds of functions, so long as the functions are differentiable. It does even more; in the form of Equation (4.3) it tells us how to construct the polynomials by giving instructions for how to compute the coefficients. Beyond that, the theorem even gives us a remainder term, a way to estimate the error in our approximations, which leads to the Lagrange error bound as described in this section. In this way, Taylor's Theorem answers the first two of the big questions that closed Section 2. And it's not done there. Later on, we will be able to use the remainder term to show that a Taylor *series* converges to the function it is supposed to model, but that's still quite a ways down the road (Sections 6 and 10).

For the most part, this concludes our discussion of Taylor polynomials. The rest of the chapter will pick up where Section 1 left off, looking at infinite series. (We will revisit Taylor polynomials and estimating error in Section 8 when we will learn a simpler method for doing so in special cases). Examining infinite series will ultimately allow us to answer the remaining two big questions from the end of Section 2, both of which involve extending our Taylor polynomials so that they go on forever. Answering these questions requires quite a few supporting ideas, so it will be a long journey. Hang in there.

Answers to Practice Problems

1. As we know, the sixth-degree Maclaurin polynomial for cos(x) is given by $P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$, so $\cos(2) \approx 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} = \frac{-19}{45}$, or -0.4222…. Now for the error. $\left|R_6(2)\right| \le \frac{M}{7!}(2-0)^7 = \frac{8}{315}M$. M is an upper bound for the largest value of the seventh derivative of $f(x) = \cos(x)$; $f^{(7)}(x) = \sin(x)$, and of course sin(x) is bounded by 1. So we use 1 for M. Our error is no more than 8/315, or about 0.025397. This means the actual value of the cos(2) must be between $-0.4\overline{2} - 0.0254$ and $-0.4\overline{2} + 0.0254$. Cranking through the arithmetic, we find that $-0.4476 \le \cos(2) \le -0.3968$. The calculator's value for cos(2) is -0.4161, which is indeed within the range we determined.

2. First we need the third-order Taylor polynomial for $f(x)$, and I don't believe we have that yet.

$$\begin{array}{cccccc} \ln x & \to & 0 & \to & 0/0! = 0 \\ x^{-1} & \to & 1 & \to & 1/1! = 1 \\ -x^{-2} & \to & -1 & \to & -1/2! = \frac{-1}{2} \\ 2x^{-3} & \to & 2 & \to & 2/3! = \frac{1}{3} \end{array}$$

Therefore $\ln(x) \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$. (Don't forget to incorporate the center into the Taylor polynomial. After not dividing by the appropriate factorial to determine the coefficients, not including the "$-a$" in $(x-a)$ is the second most common mistake in building Taylor polynomials.) Plugging in 1.5 for x gives

$$\ln(1.5) \approx (1.5-1) - \tfrac{1}{2}(1.5-1)^2 + \tfrac{1}{3}(1.5-1)^3 = 5/12 \text{ or } 0.416666\ldots.$$

$\left|R_3(1.5)\right| \le \frac{M}{4!}(1.5-1)^4 = \frac{M}{384}$, where M is a bound for the fourth derivative of the natural logarithm function for t values in the interval $[1,1.5]$. The fourth derivative is $\frac{d^4}{dt^4}\ln(t) = \frac{-6}{t^4}$. This function is negative and increasing for $t > 0$. Therefore its largest values *in magnitude* (which is to say, its smallest values) are when t is smallest, in this case when t = 1. Therefore, an appropriate value to use for M is $\left|\frac{-6}{1^4}\right| = 6$. (In this case, we are actually using the best possible value for M. That's exciting!) We conclude that $\left|R_3(1.5)\right| \le \frac{6}{384} = \frac{1}{64}$ or 0.015625. We can infer that the actual value of ln(1.5) is between $\frac{5}{12} - \frac{1}{64}$

and $\frac{5}{12}+\frac{1}{64}$. That is, we must have $0.40104 \le \ln 1.5 \le 0.43229$. Your calculator will confirm that this is indeed the case.

Example 5, parenthetical question. While we were using a higher-order Taylor polynomial, we were also trying to approximate the function at an x-value farther away from the center. Between these two competing factors (increasing the number of terms to give greater accuracy vs. moving away from the center to give lower accuracy), in this case the distance from the center won.

Section 4 Problems

1. Use the fourth-order Maclaurin polynomial for $f(x) = e^x$ to approximate *e*. Estimate the error in your approximation using the Lagrange error bound.
2. Using the fifth-order Maclaurin polynomial for $f(x) = \ln(1+x)$ to approximate $\ln(1.2)$. Estimate the error in your approximation.
3. Use a fifth-degree Maclaurin polynomial to put bounds on the value of e^2. That is, fill in the blanks: $\_\_ \le e^2 \le \_\_$.
4. Estimate the error in approximating sin(-0.3) with a 3rd-degree Maclaurin polynomial.
5. Let $f(x) = \cos\left(2x + \frac{\pi}{3}\right)$. Estimate the error in approximating (a) $f(0.1)$ and (b) $f(-0.2)$ using a 2nd-degree Maclaurin polynomial for *f*.
6. Let $f(x) = \sin x$.
   a. Estimate the error used in approximating *f*(1) using a 2nd-degree Taylor polynomial for *f* centered at $x = \frac{\pi}{3}$.
   b. If an approximation for *f*(1) that is accurate to within 10^{-9} is needed, what degree Taylor polynomial (still centered at $x = \frac{\pi}{3}$) should be used?
7. What degree Maclaurin polynomial for $f(x) = \sin x$ must be used to guarantee an approximation of sin(3) with error less than 0.0001? What if you use a Taylor polynomial centered at $x = \pi$ instead?
8. What degree Maclaurin polynomial is guaranteed to approximate cos(1) with error less than 0.0001?
9. A Taylor polynomial centered at $x = 1$ is to be used to approximate values of $f(x) = \ln x$. What degree polynomial is needed to guarantee an approximation of $\ln\left(\frac{3}{4}\right)$ with error less than 0.0001?
10. Use the second-degree Maclaurin polynomial for $f(x) = \sqrt{1+x}$ to approximate $\sqrt{1.4}$. Give bounds for your answer.
11. For what *x*-values does the approximation $\sin x \approx x - \frac{x^3}{6}$ give values that are guaranteed to be accurate to 3 decimal places (i.e., with error less than 0.0005)?
12. For what *x*-values does the approximation $\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}$ give values that are accurate to 4 decimal places?
13. Suppose the 5th-degree Maclaurin polynomial for $f(x) = \sin(x)$ is used to approximate values for $-\frac{1}{2} < x < \frac{1}{2}$. Estimate the greatest error such an approximation would produce.
14. Suppose the 2nd-degree Maclaurin polynomial for $f(x) = \ln(1+x)$ is used to approximate values for $-0.2 < x < 0.2$. Estimate the greatest error such an approximation would produce.
15. Suppose the 3rd-degree Maclaurin polynomial for $f(x) = e^x$ is used to approximate values for $-0.1 < x < 0.1$. Estimate the greatest error such an approximation would produce.
16. In Problems 8 and 9 of Section 3, you found approximations for $\sqrt{10}$ and $\sqrt[3]{10}$. Estimate the error in these approximations.
17. Suppose that $f(1) = 8$, $f'(1) = 4$, $f''(1) = -2$, and $|f'''(x)| \le 10$ for all *x* in the domain of *f*.
    a. Approximate $f(1.4)$.
    b. Estimate the error in your answer to part (a).
18. Suppose that $f(0) = 2$, $f'(0) = -3$, $f''(0) = 4$, and $|f'''(x)| \le 2$ for *x* in the interval $[-2, 2]$.
    a. Approximate $f(-1)$.
    b. Prove that $f(-1) \ne 8.75$.

19. Suppose that $g(2)=0$, $g'(2)=2$, $g''(2)=8$ and $|g'''(x)|\le 5$ for *x* in the interval $[1,3]$.
    a. Approximate $g(1.8)$.
    b. Prove that $g(1.8)$ is negative.

20. Suppose that $h(-3)=2$, $h'(-3)=5$, and $|h''(x)|\le 1$ for all *x* in the domain of *h*. Give bounds for the value of $h(-2.5)$. That is, fill in the blanks: $\_\_ \le h(-2.5) \le \_\_$.

21. Figure 4.3 shows the graph of $f^{(6)}(x)$ for some function *f*.
    a. If a fifth-degree Maclaurin polynomial is used to approximate $f(1.3)$, what is the maximum possible error in the approximation?
    b. If a fifth-degree Maclaurin polynomial is used to approximate $f(5)$, what is the maximum possible error in the approximation?
    c. If a fifth-degree Taylor polynomial centered at $x = 3$ is used to approximate $f(5)$, what is the maximum possible error in the approximation?

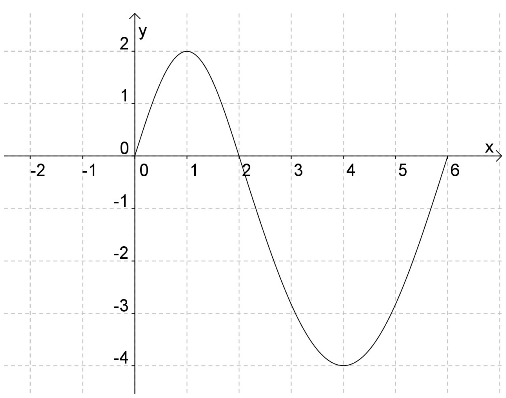

Figure 4.3: Graph of $f^{(6)}(x)$

22. Let $f(x)=\dfrac{1}{1+x^2}$.
    a. By carrying out the division $1\div\left(1+x^2\right)$, show that for this function $|R_{2n}(x)|=\dfrac{x^{2n+2}}{1+x^2}$.
    b. Conclude that $\dfrac{1}{1+x^2}=1-x^2+x^4-\cdots+(-1)^n x^{2n}+(-1)^{n+1}\cdot\dfrac{x^{2n+2}}{1+x^2}$.

23. In this problem you will explore one way to approximate the value of π and also an alternative approach to dealing with error bounds.
    a. Using the arctangent polynomial from Section 2, explain why
    $$\pi \approx 4\cdot\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\cdots+(-1)^n\cdot\tfrac{1}{2n+1}\right). \quad (4.8)$$
    b. Approximate π by using the first 5 terms of Equation (4.8). Qualitatively, how good is this approximation?
    c. Use the equation from Problem 22b to explain why $|R_{2n+1}(x)|=\int_0^x \dfrac{t^{2n+2}}{1+t^2}\,dt$ for the arctangent function.
    d. Based on part (c), explain why $|R_{2n+1}(x)|\le\int_0^x t^{2n+2}\,dt$ for $x > 0$.
    e. Carry out the integration in part (d) to obtain an explicit bound for $|R_{2n+1}(x)|$.
    f. Using your answer to part (e), determine a value of *n* that will guarantee that Equation (4.8) approximates π with error less than 0.01.

24. Show that the error in computing e^x for any positive *x*-value is no more than $\frac{3^x x^{n+1}}{(n+1)!}$ and that the error in computing e^x for any negative *x*-value is $\frac{|x|^n}{(n+1)!}$. Explain why this means that we can compute any exponential value to arbitrary precision by using a high-enough degree Maclaurin polynomial.

25. Use the Lagrange remainder to prove that all Maclaurin polynomials for $f(x)=e^x$ will underestimate the actual value of e^x for $x>0$.

26. Let $f(x)=\cos(x)$.
    a. What degree Maclaurin polynomial will be guaranteed to approximate $f(20)$

with error less than 10^{-3}? How many terms are needed in this polynomial?

b. We can reduce the answer to part (a) by exploiting the periodicity of the cosine function. Find a number t between $-\pi$ and π such that $f(t) = f(20)$. What degree Maclaurin polynomial will be guaranteed to approximate $f(t)$ with error less than 10^{-3}?

c. Using the idea from part (b), what degree Maclaurin polynomial will be guaranteed to approximate $\sin(100)$ with error less than 10^{-6}?

d. Explain why *any* sine or cosine value can be approximated with error less than 10^{-16} by using a Maclaurin polynomial of order 29. (By slightly more involved tricks, you can reduce the required degree even further, but this is good enough for us.)

27. Repeat Problem 9, but this time find the degree needed to guarantee an approximation of ln(5) with the required accuracy. Can you explain the trouble you run into? (Hint: Look at the graph of $f(x) = \ln(x)$ along with several of its Taylor polynomials centered at $x = 1$.)

28. It is a widely-held belief among calculus teachers that graphing calculators use Taylor polynomials to compute the decimal approximations you see on the screen. It turns out that this is not true, and this problem will provide some evidence for why the calculator probably *cannot* operate in this way.*

a. Determine the smallest-degree polynomial that is guaranteed to compute $e^{0.001}$ with error less than 10^{-10}. (Your calculator actually requires even greater accuracy than this, probably storing 13 to 16 decimal places in memory.)

b. Determine the smallest-degree polynomial that is guaranteed to compute e^{14} with error less than 10^{-10}.

c. If you use the degree found in part (a) to compute e^{14}, roughly how much error will there be in the approximation?

d. If you use the degree found in part (b) to compute $e^{0.001}$, roughly how much error will there be in the approximation?

(To think about: What is the calculator to do? If it always uses a polynomial with the degree you found in (a), it will not compute e^{14} with the required accuracy. But if it always uses the polynomial with the degree you found in (b), it will waste a great deal of computational time in the process, computing a value with far greater accuracy than it can actually handle with only 13-16 decimal places worth of memory storage. And remember also that your calculator is capable of accurately computing powers of e on a much larger scale. In fact, I chose the "small" number e^{14} for part (b) because my calculator could not help me answer part (b) for larger powers of e; I would hit overflow errors in trying to compute the exponential terms.*)

29. In this problem you will prove that e is irrational. The proof is by contradiction beginning with the assumption that e can be expressed as a ratio of positive integers, i.e., $e = \frac{p}{q}$. We will assume that $e < 3$. (Later on in the chapter you will be able to prove this rigorously.)

a. Pick n to be a positive integer greater than q and greater than 3. Use the n^{th}-degree Maclaurin polynomial for $f(x) = e^x$ to explain why

$$\frac{p}{q} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1). \qquad (4.9)$$

---

* I'm told that the actual algorithm used by your calculator to compute trigonometric, exponential, and logarithmic values is the CORDIC algorithm, which you can research online if you like.

* It is interesting that the problem was with the exponential pieces and not the factorials. Ultimately, factorials grow much faster than exponentials. In these computations, the factorial dominance didn't "kick in" soon enough to prevent my calculator from overflowing.

b. Multiply both sides of (4.9) by $n!$ to obtain

$$\frac{p}{q}n! = n!\left(1+\tfrac{1}{1!}+\cdots+\tfrac{1}{n!}\right)+n!\cdot R_n(1)\,. \quad (4.10)$$

Explain why both the left side of (4.10) and the first term on the right side (the $n!$ times all the stuff in parentheses) must both be integers. Explain why if we can prove that $n!\cdot R_n(1)$ is not an integer, our proof will be complete.

c. Show that $\left|R_n(1)\right| \le \dfrac{3}{(n+1)!}$.

d. Explain how it follow from part (c) that $\left|n!\cdot R_n(1)\right| \le \dfrac{3}{n+1}$. Further explain how this implies that $n!\cdot R_n(1)$ is not an integer. This completes the proof.

30. In this problem you will develop the Lagrange error bound without actually using the Lagrange form (or any other form, for that matter) of the remainder. We will assume, as always, that $f^{(n+1)}(t)$ exists and is bounded on the interval $[a,x]$.

a. Show that $f^{(n+1)}(t) = R_n^{(n+1)}(t)$ for all t in $[a,x]$ by differentiating the equation $f(x) = P_n(x) + R_n(x)$ a total of $n + 1$ times.

b. Show that if $f^{(n+1)}(t)$ is bounded by some positive number M, it follows from part (a) that $-M \le R_n^{(n+1)}(t) \le M$.

c. Take the inequality from part (b) and integrate from a to x to show that $-M\cdot(x-a) \le R_n^{(n)}(x) \le M\cdot(x-a)$. (Hint: You will need to know the value of $R_n^{(n)}(a)$. What is it and why?)

d. Show that by repeated integration the inequality in part (c) can ultimately be expressed as

$$\frac{-M}{(n+1)!}(x-a)^{n+1} \le R_n(x) \le \frac{M}{(n+1)!}(x-a)^{n+1}.$$

e. Conclude that $\left|R_n(x)\right| \le \dfrac{M}{(n+1)!}\left|x-a\right|^n$, in agreement with Theorem 4.2.

Section 5 – Another Look at Taylor Polynomials (Optional)

In Section 3, we showed how to build a Taylor polynomial by matching the derivatives of the polynomial to the derivatives of the function being modeled. Then in Section 4 we extended the MVT (a significant theoretical result of differential calculus) to the Lagrange form of the remainder term so that we could estimate the error in our polynomial approximations.

In this section we give an alternate development of Taylor polynomials. The difference will be that instead of looking at derivatives at a single point, we will look at integrals on an interval. One neat feature of this approach is that the remainder term arises naturally, almost unasked for.

For this entire section, we will be working on the interval bounded by a number a and a number x (i.e., either $[a,x]$ or $[x,a]$). We will use t as a dummy variable for integration. Our trick will be to use a lot of integration by parts.

Rebuilding the Taylor Polynomial

We begin with

$$f(x) = f(a) + \int_a^x f'(t)\,dt. \tag{5.1}$$

Note that this is really just the Fundamental Theorem. If you don't recognize it as such, swing the $f(a)$ over to the other side. Then you have $f(x) - f(a) = \int_a^x f'(t)dt$ which is definitely the Fundamental Theorem. The version of the Fundamental Theorem in Equation (5.1) is useful for thinking of definite integrals as accumulators of change, but that's not really relevant to the task at hand.

We will carry out the integration in Equation (5.1) via integration by parts according to the scheme

$$\begin{array}{ll} u = f'(t) & dv = dt \\ du = f''(t)dt & v = t - x. \end{array}$$

We've been a little crafty here. Normally you would say that if $dv = dt$, then $v = t$. But really it follows that $v = t + C$. Usually when we do integration by parts we leave the $+C$ for the end. If we want, though, we can incorporate it into the substitution scheme. Since our variable of integration is t, x is a constant with respect to the integration. Therefore, the $-x$ amounts to the same thing as a $+C$. In a moment you will see why this is a good idea.

Carrying out the integration gives

$$\begin{aligned} f(x) &= f(a) + f'(t)(t-x)\Big|_a^x - \int_a^x (t-x) f''(t)\,dt \\ &= f(a) + f'(x)(x-x) - f'(a)(a-x) - \int_a^x (t-x) f''(t)\,dt. \end{aligned}$$

Cancelling and cleaning up some signs gives

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x (x-t) f''(t)\,dt. \tag{5.2}$$

The first two terms in the right side of (5.2) are, of course, the first-order Taylor polynomial for $f(x)$.

Let's attack the integral in (5.2) with another round of integration by parts. The substitution scheme will be similar, though we won't need to incorporate a constant of integration explicitly this time. We will let

$$u = f''(t) \qquad dv = (x-t)dt$$
$$du = f'''(t)dt \quad v = \frac{-(x-t)^2}{2}.$$

Remember that our variable of integration is *t*; this is why there's a negative sign in the expression for *v*. Carrying out the integration by parts, we have

$$f(x) = f(a) + f'(a)(x-a) + f''(t)\cdot\left.\frac{-(x-t)^2}{2}\right|_a^x - \int_a^x \frac{-(x-t)^2}{2}\cdot f'''(t)\,dt.$$

When we evaluate that middle term at *x*, it will vanish, as it did in the previous iteration. So let us press on and clean up the signs to obtain

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\cdot\frac{(x-a)^2}{2} + \int_a^x \frac{(x-t)^2}{2}\cdot f'''(t)\,dt. \tag{5.3}$$

Notice that we have now created the second-order Taylor polynomial, pretty much out of nothing.

Let's do it one more time, just to make sure you see how this is working. We will apply integration by parts to the integral in (5.3) according to the scheme

$$u = f'''(t) \qquad dv = \frac{(x-t)^2}{2}dt$$
$$du = f^{(4)}(t)dt \qquad v = \frac{-(x-t)^3}{6}.$$

This yields

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + f'''(t)\cdot\left.\frac{-(x-t)^3}{6}\right|_a^x - \int_a^x \frac{-(x-t)^3}{6}\cdot f^{(4)}(t)\,dt$$

or

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \int_a^x \frac{(x-t)^3}{6} f^{(4)}(t)\,dt\,. \tag{5.4}$$

One can show using mathematical induction that what we have observed so far applies to any positive integer *n*. That is, if f is differentiable $n+1$ times, then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x \frac{(x-t)^n}{n!}\cdot f^{(n+1)}(t)\,dt\,. \tag{5.5}$$

The first $n+1$ terms on the right side of Equation (5.5) are, of course, $P_n(x)$. We have recovered our Taylor polynomial from Section 3, and this time it seemed to build itself! As a by product, we have a remaining term in (5.5). This must be the remainder term, and so we have a new version of $R_n(x)$.

$$R_n(x) = \int_a^x \frac{(x-t)^n}{n!}\cdot f^{(n+1)}(t)\,dt \tag{5.6}$$

The form of the remainder term in Equation (5.6) is called, appropriately enough, the integral form of the remainder. (There are some technical considerations about the integral existing, but if we assume that $f^{(n+1)}$ is continuous, then we're safe.)

Playing with the Integral Form of the Remainder

The use of the integral form of the remainder is not necessarily in computing exact values of the integral. This was not ultimately our goal with the Lagrange form either. However, one interesting feature of the integral form is that it can give us the Lagrange error bound directly.

Let us suppose, as we did in Section 4, that $f^{(n+1)}(t)$ is bounded on $[a, x]$, i.e., we assume that there is a positive number M such that $\left|f^{(n+1)}(t)\right| \le M$ for all t in the interval. This means that since

$$R_n(x) = \int_a^x \frac{(x-t)^n}{n!} \cdot f^{(n+1)}(t)\,dt\,,$$

it follows that

$$\left|R_n(x)\right| \le \int_a^x \frac{(x-t)^n}{n!} \cdot M\,dt \tag{5.7}$$

Let's play with the right side of this inequality a bit.

$$\begin{aligned}\int_a^x \frac{(x-t)^n}{n!} \cdot M\,dt &= M\int_a^x \frac{(x-t)^n}{n!}dt \\ &= M \cdot \left.\frac{-(x-t)^{n+1}}{(n+1)!}\right|_a^x \\ &= M \cdot \frac{(x-a)^{n+1}}{(n+1)!}\end{aligned}$$

Combining this result with Inequality (5.7), we obtain our old friend the Lagrange error bound:

$$\left|R_n(x)\right| \le M \cdot \frac{(x-a)^{n+1}}{(n+1)!}.$$

So we see that there is nothing particularly "Lagrangian" about the Lagrange error bound.

In fact, not only can we get the Lagrange error bound from the integral form of the remainder, we can obtain the explicit Lagrange *remainder* from the integral form as well. Doing so requires a theorem that is left out of many first-year calculus books.

Theorem 5.1 – Generalized Mean Value Theorem for Integrals
If f is continuous on $[a,b]$ and g is integrable and non-negative (or non-positive) on $[a,b]$, then there exists a number z in $[a,b]$ such that

$$f(z) \cdot \int_a^b g(x)dx = \int_a^b f(x)g(x)dx.$$

The "regular" MVT for Integrals basically says that a continuous function will obtain its average value at some point. If you think of the function g in Theorem 5.1 as being some kind of weighting function, then this theorem says that a continuous function f obtains its "weighted average" at some point on the interval. The details are not too important for us. The point is to apply this to the integral form of the remainder.

$$R_n(x) = \int_a^x \frac{(x-t)^n}{n!} \cdot f^{(n+1)}(t)\,dt = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$$

Let $(x-t)^n$ take the role of $g(x)$ and $f^{(n+1)}(t)$ to play the role of $f(x)$. Then Theorem 5.1 tells us that there is a number z such that

$$\begin{aligned} R_n(x) &= \frac{f^{(n+1)}(z)}{n!}\int_a^x (x-t)^n\, dt \\ &= \frac{f^{(n+1)}(z)}{n!}\cdot \left.\frac{-(x-t)^{n+1}}{n+1}\right|_a^x \\ &= \frac{f^{(n+1)}(z)}{(n+1)!}\cdot (x-a)^{n+1}. \end{aligned}$$

And poof! There's our Lagrange form of the remainder. And there are yet more forms of the remainder term. You can look them up online if you are interested.

I am still partial to the development of Taylor polynomials presented in Section 4. Matching the derivatives makes a lot of sense to me as a way to create a polynomial that will model a given function. But I also think it is truly fascinating how the integration by parts approach builds the Taylor polynomial automatically; we create it without any conscious desire or plan to do so. And yet it works so smoothly. To me, this is one of the things that makes mathematics cool.

Section 6 – Power Series and the Ratio Test

From Taylor Polynomials to Taylor Series

We are about to make a major shift in our focus. In this section we move from polynomials (with finitely many terms) to power series, polynomial-like functions that have infinitely many terms. In terms of modeling functions, this is the leap from Taylor *polynomials*, which are only approximate representations for functions, to Taylor *series*, a way of representing functions exactly. The Taylor series is more than an approximation for the function being modeled; it *is* the function… usually. There are some exceptions and caveats, but they are down the road a bit.

This is a big conceptual leap. If you are uncertain about it, you are right to be a little cautious. The infinite can be a strange thing. But let the picture convince you.

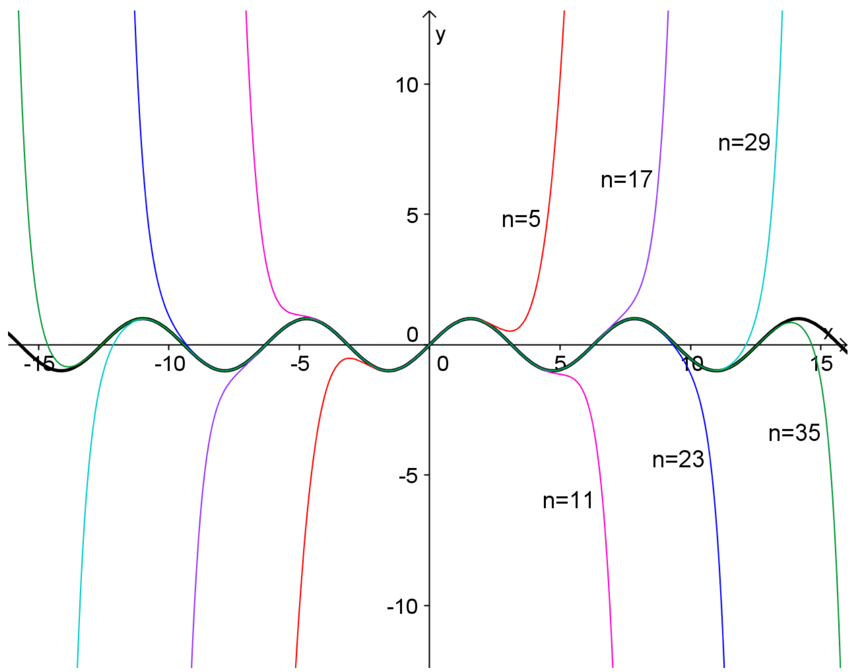

Figure 6.1: Maclaurin polynomials for $f(x)=\sin(x)$

Figure 6.1 shows Macluarin polynomials of various degrees (the indicated n-values) for the sine function. It does not show you anything new or surprising; it just collects several pictures into one place. The figure shows, as we have seen, that as we add more terms to the polynomial we get a better fit on a wider interval. In fact, we have seen the pattern that generates these polynomials:

$$f(x)\approx\sum_{k=0}^{n}(-1)^k\cdot\frac{x^{2k+1}}{(2k+1)!}.$$

Is it such a stretch to imagine that with infinitely many terms we could make the error go to zero for all x? Can we not imagine that

$$\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n+1}}{(2n+1)!}$$

would match the sine function exactly? The claim here is that while we know that $f(x)$ is approximately given by $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\cdot\frac{x^{2n+1}}{(2n+1)!}$, perhaps it is given exactly by $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\cdot\frac{x^{2n+1}}{(2n+1)!}+\cdots$, with the extra "$+\cdots$" at the end making all the difference.

As another example, take $g(x) = \frac{1}{1-x}$ (Figure 6.2). The polynomials are of the form $P_n(x) = 1 + x + x^2 + \cdots + x^n$. But earlier in the chapter we expanded this as a geometric series. We *already* know, in some sense, that $g(x) = \sum_{n=0}^{\infty} x^n$.

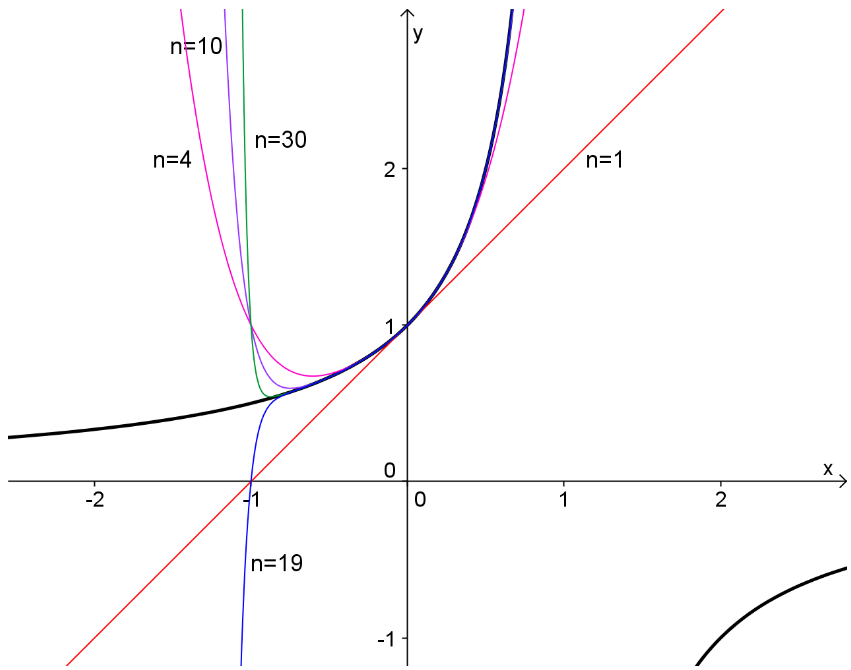

Figure 6.2: Maclaurin polynomials for $g(x) = \frac{1}{1-x}$

The picture, though, is different from what we saw in Figure 6.1. When we graph several partial sums of the infinite geometric series $1 + x + x^2 + \cdots$ (i.e., Taylor polynomials), we do not see the fit extend forever. For one thing, no matter how high we push the degree, no polynomial seems to be able to model the unbounded behavior in the graph of g near $x = 1$. The underlying issue here is of convergence. The common ratio of the geometric series $1 + x + x^2 + \cdots$ is x, so this series does not converge if $|x| \geq 1$; the series just doesn't make sense for such x. The high-order Taylor polynomials in Figure 6.2 are trying to show us how the series breaks down outside the interval $(-1,1)$. We can't really graph the infinite series because there are infinitely many terms. If we could it would look like Figure 6.3.

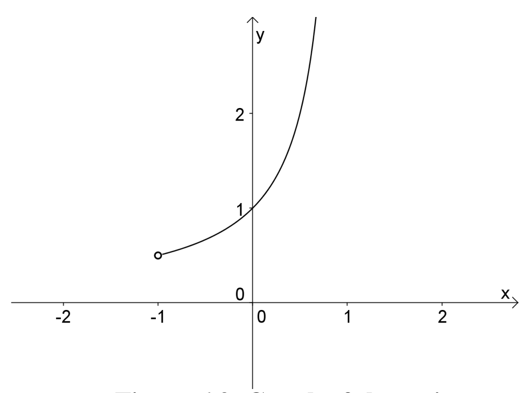

Figure 6.3: Graph of the series $g(x) = 1 + x + x^2 + \cdots$

As a final example, consider $h(x) = \sqrt{x}$, with Taylor polynomials centered at $x = 4$. Like with the earlier example of $f(x)$, we can build polynomials, see the pattern in the coefficients, and extrapolate to an infinite series.* But like the example of $g(x)$, the graphs of the partial sums (the Taylor polynomials) suggest a convergence issue (see Figure 6.4). It appears that the series converges from $x = 0$ to about $x = 8$ because that is where the higher-order polynomials appear to fit the function well. However, this series does not happen to be geometric, so we can't really be sure as of yet. We do not yet have the tools to deal with convergence of non-geometric series. In order to establish the convergence of a series like the one representing the square root function, we need to talk about power series, and that topic will dominate most of the remainder of this chapter.

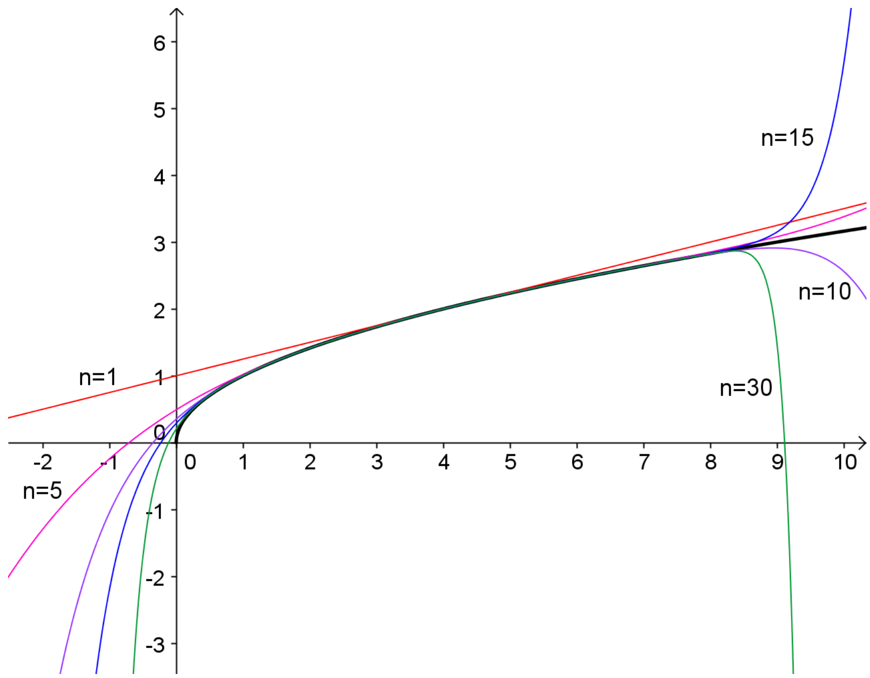

Figure 6.4: Taylor polynomials for $h(x) = \sqrt{x}$ centered at $x = 4$

Power Series

We start right off with the definition.

Definition	A **power series centered at $x = a$** is an infinite series of the form $$\sum_{n=0}^{\infty} c_n (x-a)^n \qquad (6.1)$$ We also allow the initial value of the index to be a positive integer (instead of zero).

Thus a power series is an infinite series like those seen in Section 1, but it is also a function of x. Notice that if $a = 0$, the series takes on the relatively simple form $\sum_{n=0}^{\infty} c_n x^n$. The c_ns are just numbers. (The c stands for coefficient.) The coefficients can have a pattern which can depend on n, as in

* I'll let you work out the details for yourself if you like. This function can be represented as $2 + \sum_{n=1}^{\infty} (-1)^{k+1} \frac{(2n-3)!!}{n! \cdot 2^{3n-1}} (x-4)^n$, where !! means the "double factorial." The double factorial is like the regular factorial, except that you skip every other factor. $5!! = 5 \times 3 \times 1$, $6!! = 6 \times 4 \times 2$, and by definition 0!! = (-1)!! = 1.

$$1(x-3)+4(x-3)^2+9(x-3)^3+16(x-3)^4\cdots=\sum_{n=1}^{\infty}n^2(x-3)^n$$

or

$$1+1x+2x^2+6x^3+24x^4+\cdots=\sum_{n=0}^{\infty}n!x^n .$$

Or the coefficients can be random (or random-appearing), as in

$$3+1x+4x^2+1x^3+5x^4+9x^5+\cdots.^*$$

Some of the coefficients can also be zero, which accounts for missing terms in a series like the one for the sine function.

There is one notational hole we have to plug up with our definition of a power series. If $f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n$, then $f(a)$ is technically undefined as it results in the operation 0^0. But this is clearly not what we mean by the power series. The sigma notation is just a shorthand for writing $f(x)=c_0+c_1(x-a)+c_2(x-a)^2+\cdots$. In this form, it is clear that $f(a)=c_0$. Despite the notational glitch, we will understand that a power series evaluated at its center is always c_0 and never undefined.

The previous paragraph points out that every power series converges at its center. But for what other x-values does a given power series converge? As with any infinite series, convergence is an issue with power series. In Section 1, a given series either converged or diverged. That was the whole story. *Power* series, however, are more complicated because they depend on x. A particular power series may converge for some values of x but not for others. The set of x-values for which the power series converges is called the **interval of convergence**. One nice feature about power series is that the sets of x-values for which they converge are fairly simple. They always converge on a (sometimes trivial) interval, and the center of the series is always right in the middle of that interval. The distance from the center of this interval to either endpoint is called the **radius of convergence** of the series, often denoted R. It turns out that there are only three possible cases that can come up.

1. The "bad" case is where $R = 0$. A series with $R = 0$ converges only at its center. An example of such a series is $\sum_{n=0}^{\infty}n!x^n$.
2. The "best" case is where $R=\infty$. By this we mean that the radius of convergence is infinite; the series converges for *all* x-values. As we will show, this is the case for the series $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$ which represents the sine function.
3. The other case is where R is a positive, finite number. In this case, the series converges for $|x-a|<R$ and diverges for $|x-a|>R$. In other words, the series converges for x-values within R units of the center of the series but diverges for x-values that are more than R units from the center. (An alternate symbolic expression of this idea is to say that the series converges if $a-R<x<a+R$ and diverges if either $x<a-R$ or $x>a+R$.) What happens if $|x-a|=R$ depends on the particular series. It will take most of the rest of this chapter to answer the question of convergence at the endpoints of the interval. Both $g(x)$ and $h(x)$ from earlier in this section are examples of this last case. For $g(x)$, we have $a = 0$ and $R = 1$; the series converges for all x within one unit of $x = 0$ and diverges otherwise. We knew this from our work with geometric series. For $h(x)$, the center is $a = 4$. It appears from the graphs of partial sums for the series that

* These coefficients here aren't really random, but I've used a common trick for making random-seeming numbers. Can you tell what it is?

$R = 4$, meaning that the series converges for $0 < x < 8$. It is unclear at this point whether the series converges at $x = 0$ or at $x = 8$, so we cannot yet write the interval of convergence. (It could be $(0,8)$, $[0,8)$, $(0,8]$, or $[0,8]$.)

To state one more time: All power series converge at their centers, even the "bad" ones. Also, just so we are clear about what we are discussing, note that you can have series involving *x*s that are not power series. The series $\sum (\sin x)^n$ and $\sum \frac{1}{x^n}$ are examples of series that are not power series. We do not call them power series because they are not like polynomials; they do not consist of positive integer powers for *x*. But even a series like $\sum \left(x^2 - 5\right)^n$, which is "polynomial-like," is not a power series because it cannot be written in the form presented in Equation (6.1) in the definition. Series that are not power series can have much more complicated convergence properties; they need not fall into one of the three cases listed above, and indeed none of the examples in this paragraph does.

Example 1

The power series $\sum_{n=0}^{\infty} c_n x^n$ converges for $x = 3$ and diverges for $x = -5$. For each of the following *x*-values, state whether the series converges, diverges, or if the convergence at that point cannot be determined: -6, -4, -3, -2, 0, 2, 4, 5, and 6.

Solution

First, observe that the center of this series is $x = 0$. Since the series converges for $x = 3$, which is 3 units away from the center, the radius of convergence must be at least 3. Since the series diverges at $x = -5$, 5 units from the center, *R* can be no more than 5. This means the series converges for *x*-values that are less than 3 units from the center and diverges for all *x*-values more than 5 units from the center.

The series converges at $x = -2$, 0, and 2. (Of course, $x = 0$ was a freebie. Every series converges at its center.)

We can draw no conclusion about convergence at $x = -4$, -3, or 5. The radius of convergence is somewhere between 3 and 5, and this puts -4 squarely in the no-man's land; it might be within the radius of convergence from the center or it might be outside of it. We cannot tell without more information. As for $x = -3$ and $x = 5$, they could be exactly *R* units away from the center since we might have $R = 3$ or $R = 5$ (or anything in between). At an *x*-value that is *R* units from the center, a series may converge or may diverge. We do not know which without more information.

The series diverges at $x = 6$ and $x = -6$ since both these points are more than 5 units away from the center of the series. ◊

Practice 1

The power series $\sum_{n=0}^{\infty} c_n (x-2)^n$ converges at $x = 6$ and diverges at $x = 8$. For each of the following *x*-values, state whether the series converges, diverges, or if the convergence at that point cannot be determined: -8, -6, -2, -1, 0, 2, 5, 7, and 9.

We saw in Section 2 that term-by-term calculus operations produced new Taylor polynomials from old ones. This is true of power series as well. We can differentiate or integrate a power series, one term at a time. This is a very convenient way to create new power series or analyze existing ones. One fact that makes this kind of manipulation simple is that *term-by-term operations do not change the radius of convergence of a series*. That is, *R* is unchanged when you differentiate or integrate a power series. For example, we know that $1 - x + x^2 - \cdots + (-x)^n + \cdots = \frac{1}{1+x}$ converges with $R = 1$. Integrating term by term gives $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \cdot \frac{x^n}{n} + \cdots = \ln(1+x)$ which must *also* converge with $R = 1$. This latter series definitely converges for $-1 < x < 1$. Whether it converges at the endpoints is trickier. It turns out that when we integrate a series, endpoints that did not initially converge may converge in the new series. The opposite is true of differentiation; endpoints that did converge might be lost. There is no way to predict this; you will just have to check using the tools to be presented in Sections 7 and 8.

Example 2

Find the radius of convergence of the power series $f(x) = \sum_{n=0}^{\infty}\left(\frac{x-3}{2}\right)^n$ and $g(x) = \sum_{n=1}^{\infty}\frac{n}{2^n}(x-3)^{n-1}$.

Solution

The series defining $f(x)$ is geometric with common ratio $r = \frac{x-3}{2}$. As we know, a geometric series converges if and only if $|r| < 1$, so we must have $\left|\frac{x-3}{2}\right| < 1$ if the series for $f(x)$ is to converge. Then we have $\left|\frac{x-3}{2}\right| < 1 \Rightarrow \frac{|x-3|}{2} < 1 \Rightarrow |x-3| < 2$ which says that *x*-values must be within 2 units of $x = 3$. Hence, the radius of convergence is 2. If you are uncomfortable interpreting absolute value inequalities in terms of distance, now would be a good time to become more comfortable. In the meantime, though, $|x-3| < 2$ is the same as $-2 < x-3 < 2$. Equivalently, $1 < x < 5$. This interval is 4 units wide, so its radius must be 2.

The series defining $g(x)$ is not geometric. Fortunately, if you write out some terms of both *f* and *g*, you will see that *g* is the derivative of *f*. (Remember: When in doubt, write out a few terms!) Since *g* is the derivative of *f*, its power series must have the same radius of convergence as *f*, namely 2. ◊

The Ratio Test

In Example 2 we found the radius of convergence of a geometric power series by using the fact that $|r|$ must be less than 1 for a geometric series to converge. If a series is not geometric, and we cannot easily relate it to one by differentiation or integration, we are out of luck. We need a new tool, and the best one in the shop is the **ratio test**. We'll state the test, explain it, and then give a few examples.

Theorem 6.1 – The Ratio Test

Let $\sum a_n$ be a series in which $a_n > 0$ for all *n* (or at least all *n* past some particular threshold value *N*).

Form the ratio $\frac{a_{n+1}}{a_n}$ and evaluate its limit as $n \to \infty$. Provided this limit exists, there are three possible cases.

If $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} > 1$, then $\sum a_n$ diverges. (As a bonus, we include $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \infty$ in this case.)

If $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} < 1$, then $\sum a_n$ converges.

If $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1$, then the ratio test is inconclusive. $\sum a_n$ could either converge or diverge; another tool is needed to decide the series.

The idea behind the ratio test is to see if, long term, the given series behaves like a geometric series. For a geometric series, $\frac{a_{n+1}}{a_n}$ is constant—the common ratio of the series—and only if $\left|\frac{a_{n+1}}{a_n}\right| < 1$ do the terms shrink fast enough for the series to converge. In the ratio test, we no longer suppose that the ratio of successive terms is constant, but we are saying that if that ratio eventually gets—and stays—below 1, then the terms will decrease at a rate on par with a convergent geometric series. Recall that the terms of a convergent geometric series go to zero *fast*. If we can show via the ratio test that a given series has terms going to zero this quickly, then that will establish the convergence of the series.

Example 3

Use the ratio test to determine whether the following series converge.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n}$ c. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n}}$

Solution

a. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$: For this series $a_n = \frac{1}{n!}$, so $a_{n+1} = \frac{1}{(n+1)!}$. We must evaluate the limit $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}$.

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

0 < 1, so this series converges. (Compute some partial sums. Can you guess to what value the series converges?)

b. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n}$: For this series $a_n = \frac{3^n}{n}$ and $a_{n+1} = \frac{3^{n+1}}{n+1}$. We must evaluate $\displaystyle\lim_{n\to\infty} \frac{3^{n+1}/(n+1)}{3^n/n} = \lim_{n\to\infty} \frac{n \cdot 3^{n+1}}{(n+1)\cdot 3^n}$.

$$\lim_{n\to\infty} \frac{n \cdot 3^{n+1}}{(n+1)\cdot 3^n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\cdot 3\right) = 3$$

3 > 1, so this series diverges. Quickly. In the long run, the terms are growing roughly as fast as a geometric series with common ratio 3. In fact, we should never have used the ratio test on this series. It doesn't even pass the n^{th} term test ($\lim_{n\to\infty} a_n \neq 0$). Convergence was never a possibility.

c. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n}}$: $a_n = \frac{1}{\sqrt{2n}}$ and $a_{n+1} = \frac{1}{\sqrt{2n+2}}$. Be careful with a_{n+1}. When we plug in $n+1$ in the place of *n*, we must distribute the 2, which is why we end up with $2n+2$ instead of $2n+1$. Not distributing the 2 is a common error that you should watch for.

$$\lim_{n\to\infty} \frac{1/\sqrt{2n+2}}{1/\sqrt{2n}} = \lim_{n\to\infty} \frac{\sqrt{2n}}{\sqrt{2n+2}} = \lim_{n\to\infty} \sqrt{\frac{2n}{2n+2}} = 1$$

Because the limit is equal to 1, the ratio test has failed us. We cannot determine whether this series converges or diverges without additional convergence tests. (Between you and me, it diverges.) ◊

Example 3 shows some of the strengths and weaknesses of the ratio test. It works very well on series whose terms involve factorials and exponential factors because of the way these types of things cancel when put into the fraction a_{n+1} / a_n. When you see a factorial, you should almost always think ratio test. For power terms, like in the third series (which is essentially like $n^{-1/2}$), the ratio test is typically inconclusive.

Practice 2

Use the ratio test to determine whether the following series converge.

a. $\sum_{n=1}^{\infty} \frac{n}{(n+1)^2}$ b. $\sum_{n=0}^{\infty} (2n)!$ c. $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

The series in Example 3 were series of *constants*. There were no xs in those series. The point of the ratio test was to help with *power series*, so we turn our attention to them now. There is a snag, though. Consider the power series $\sum_{n=0}^{\infty} \frac{(x-3)^n}{2n}$. For this series $a_n = \frac{(x-3)^n}{2n}$, but this means that a_n is not necessarily positive. This is bad because one of the hypotheses of the ratio test is that $a_n > 0$ for all n.

It turns out that it is good enough (in fact, in some ways even better) to look at the absolute values of the terms in the series. Instead of applying the test to $a_n = \frac{(x-3)^n}{2n}$, we will apply it to $|a_n| = \left|\frac{(x-3)^n}{2n}\right|$ which is always non-negative. In Sections 8 and 9 we will talk in great detail about the difference between the convergence of $\sum a_n$ and $\sum |a_n|$, but it would be a real shame to delay our study of power series until then.

Example 4

Determine the radius of convergence for the power series $\sum_{n=0}^{\infty} \frac{(x-3)^n}{2n}$.

Solution

As we will do with all non-geometric power series, we apply the ratio test to the absolute value of the terms: $|a_n| = \left|\frac{(x-3)^n}{2n}\right|$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{\left|\frac{(x-3)^{n+1}}{2(n+1)}\right|}{\left|\frac{(x-3)^n}{2n}\right|} \\
&= \lim_{n\to\infty} \left|\frac{2n\cdot(x-3)^{n+1}}{(2n+2)\cdot(x-3)^n}\right| \\
&= \lim_{n\to\infty} \left(\frac{2n}{2n+2}\cdot|x-3|\right) \\
&= |x-3|
\end{aligned}$$

According to the ratio test, in order for the series to converge, this limit must be less than 1. That is, we have convergence when $|x-3|<1$. From here, we can infer that the radius of convergence for this series is $R = 1$. ($|x-3|<1 \Rightarrow -1 < x-3 < 1 \Rightarrow 2 < x < 4$) Furthermore, we know that the series will converge when x is in the interval $(2,4)$ and diverge when $x > 4$ or $x < 2$. If $x = 2$ or $x = 4$, we can conclude nothing at this point. ◊

Practice 3

Determine the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{nx^n}{4^n}$.

Please do take a moment to recognize that there are two different ways that we are using the ratio test in this section. In Example 3, we used the ratio test to determine the convergence of a series of constants. This is good practice, and we can learn a lot about how the ratio test works from such problems. But our ultimate objective was to use the ratio test for power series like in Example 4 and Practice 3. These are slightly different in flavor, and I wanted to call explicit attention to the difference so as to avoid confusion.

Updating Our Understanding of Modeling Functions

Let's take a moment to go back to the four big questions from the end of Section 2 and see if we have gotten anywhere.

As we know, we answered Question 1 (about building Taylor polynomials) and Question 2 (about error estimation) in the first half of the chapter. But this leaves the last two questions:

3. *When can we extend the interval on which the Taylor polynomial is a good fit indefinitely?*
4. *Can we match a function perfectly if we use infinitely many terms? Would that be meaningful?*

We'll take Question 4 first, as looking at it carefully will help us better understand how to think about Question 3. Question 4 can be considered from a graphical perspective as well an error estimation perspective. On the graphical side, the question asks whether we can model a function's graph perfectly if we use infinitely many terms. This is essentially the question that opened this section. But another entry point for Question 4 is to ask what happens to the error of the polynomial approximation as we include more and more terms. Can we in fact drive the error to zero by including infinitely many terms?

Let's take everything we know so far and grapple with these ideas as they pertain to $f(x)=\sin(x)$. We know from Section 2 that $f(x)\approx\sum_{k=0}^{n}(-1)^k\cdot\frac{x^{2k+1}}{(2k+1)!}$. The claim we made at the beginning of this section was that equality could be achieved by letting that upper limit of summation go to infinity and converting the polynomial into a power series: $f(x)=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n+1}}{(2n+1)!}$. The second part of Question 4 tells us to ask if this is even meaningful. We now know how to answer that; we find out where this series converges by applying the ratio test.

$$\lim_{n\to\infty}\frac{\left|(-1)^{n+1}\cdot\frac{x^{2n+3}}{(2n+3)!}\right|}{\left|(-1)^n\cdot\frac{x^{2n+1}}{(2n+1)!}\right|}=\lim_{n\to\infty}\left|\frac{(2n+1)!}{(2n+3)!}\cdot\frac{x^{2n+3}}{x^{2n+1}}\right|=\lim_{n\to\infty}\frac{x^2}{(2n+3)(2n+2)}$$

Remember that we are taking the limit as $\boldsymbol{n}$ goes to infinity, not x. For any fixed x-value, this limit is 0, which is less than 1. This power series converges for all x. It turns out that this is also the best way to look at Question 3. We always suspected that the sine function could be modeled on an arbitrarily wide interval; the convergence of its power series for all x is the reason why this is indeed the case.

But we have not really addressed Question 4 yet. Sure, the power series converges. But is it a perfect match for the sine function? In a word, yes. Take a look at the Lagrange error bound. Since the power series is centered at the origin, the error bound takes the form

$$|R_n(x)|\le\frac{M}{(n+1)!}\cdot|x|^{n+1}.$$

$f(x)$ and all its derivatives are bounded by 1, so in this case the error bound takes the simpler form

$$|R_n(x)|\le\frac{|x|^{n+1}}{(n+1)!}.$$

Now consider what happens to our error estimate as the number of terms (n) is allowed to go to infinity. $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ for any x because the factorial denominator will eventually dominate the exponential numerator no matter how large the base. We see then that the error in "estimating" the sine function with an infinitely long "polynomial"—a power series—is zero for all x. Indeed, this power series *does* match the sine function perfectly because there is no error.

We will take one more example before leaving this section and letting these ideas percolate until Section 10. Consider $g(x) = \ln(1+x)$. We know from Section 3 that we can approximate g with a Maclaurin polynomial: $g(x) \approx x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + (-1)^{n+1} \cdot \frac{1}{n} x^n = \sum_{k=1}^{n} (-1)^{k+1} \cdot \frac{x^k}{k}$. The claim is that the power series $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n}$ is a prefect match for $g(x)$. As we did with $f(x)$, let's look at the radius of convergence by applying the ratio test.

$$\lim_{n\to\infty} \frac{\left|(-1)^{n+2} \cdot \frac{x^{n+1}}{n+1}\right|}{\left|(-1)^{n+1} \cdot \frac{x^n}{n}\right|} = \lim_{n\to\infty} \left| \frac{n}{n+1} \cdot \frac{x^{n+1}}{x^n} \right| = \lim_{n\to\infty} \frac{n}{n+1} \cdot |x| = |x|$$

The ratio test requires that this limit be less than 1 for convergence, so we set $|x| < 1$ which implies that the power series converges for $-1 < x < 1$. We do not yet have enough tools to determine what happens when $x = \pm 1$, but we know for sure that the power series diverges if $|x| > 1$. This means that the series is no good for evaluating, for example, ln(5). (Compare this to the results of Problem 27 from Section 4 in which you tried to estimate the error in an approximation of ln(5) and found that the error could not be controlled.) However, for evaluating $\ln\left(\frac{1}{2}\right)$, which is $\ln\left(1 + \frac{-1}{2}\right)$, the power series should in fact do quite nicely. Again, looking at the radius of convergence of the power series answers Question 3 for us, at least mostly; we still don't know about the endpoints of the interval of convergence, but we at least know why not all functions can be represented on arbitrarily wide intervals; the intervals of convergence of their power series are not large enough.

As you can see, we are quite close to being able to answer both Questions 3 and 4. We still need a few more tools for determining convergence at endpoints, but the significant ideas are already in place. Sections 7-9 will be a bit of a digression as we build up a set of tools for deciding endpoint convergence, but don't forget the big ideas about power series from this section as you work through the coming convergence tests.

Answers to Practice Problems

1. The center of this series is 2 and the radius of convergence is between 4 (6 is 4 units away from 2) and 6 (8 is 6 units from 2).
The series converges at x = -1, 0, 2 (the center), and 5. These points are within 4 units of 2.
The series diverges at x = -8, -6 and 9. These points are more than 6 units away from 2.
No conclusion can be drawn for x = -2 or 7. Seven is 5 units from the center, so it is in no-man's land. $x = -2$ is a potential endpoint (it is 4 units from the center), so we cannot tell if the series converges there or not.

2. a. $\sum_{n=1}^{\infty} \frac{n}{(n+1)^2}$: $a_n = \frac{n}{(n+1)^2}$, so $a_{n+1} = \frac{n+1}{\left((n+1)+1\right)^2} = \frac{n+1}{(n+2)^2}$. We must evaluate $\lim_{n\to\infty} \frac{\frac{n+1}{(n+2)^2}}{\frac{n}{(n+1)^2}}$.

$$\lim_{n\to\infty}\frac{\frac{n+1}{(n+2)^2}}{\frac{n}{(n+1)^2}}=\lim_{n\to\infty}\frac{(n+1)(n+1)^2}{n(n+2)^2}=1$$

Unfortunately, the ratio test is inconclusive here.

b. $\sum_{n=0}^{\infty}(2n)!$: For the moment, let's ignore the fact that this series spectacularly fails the n^{th} term test and must therefore diverge. We'll use the ratio test as we're told. $a_n=(2n)!$, so $a_{n+1}=\left(2(n+1)\right)!=(2n+2)!$.

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(2n+2)!}{(2n)!}=\lim_{n\to\infty}\frac{(2n+2)(2n+1)(2n)(2n-1)\cdots 2\cdot 1}{(2n)(2n-1)\cdots 2\cdot 1}=\lim_{n\to\infty}(2n+2)(2n+1)=\infty$$

As expected, the limit is greater than 1, indicating that the series diverges.

c. $\sum_{n=0}^{\infty}\frac{2^n}{n!}$: $a_n=\frac{2^n}{n!}$, so $a_{n+1}=\frac{2^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}=\lim_{n\to\infty}\frac{2^{n+1}\cdot n!}{2^n\cdot(n+1)!}=\lim_{n\to\infty}\frac{2}{n+1}=0$$

This limit is less than 1, so the series converges by the ratio test.

3. We begin by applying the ratio test to $\sum\left|a_n\right|$:

$$\begin{aligned}\lim_{n\to\infty}\frac{\left|a_{n+1}\right|}{\left|a_n\right|}&=\lim_{n\to\infty}\frac{\left|\frac{(n+1)x^{n+1}}{4^{n+1}}\right|}{\left|\frac{nx^n}{4^n}\right|}\\&=\lim_{n\to\infty}\left|\frac{(n+1)\cdot 4^n\cdot x^{n+1}}{n\cdot 4^{n+1}\cdot x^n}\right|\\&=\lim_{n\to\infty}\left(\frac{n+1}{n}\cdot\frac{1}{4}\cdot\left|x\right|\right)\\&=\frac{\left|x\right|}{4}\end{aligned}$$

Now we need to have $\frac{|x|}{4}<1$ for convergence, or equivalently $\left|x\right|<4$. The radius of convergence of this series is $R = 4$.

Section 6 Problems

In Problems 1-12, use the ratio test to determine whether the series converges or diverges. If the ratio test is inconclusive, state that as well.

1. $\frac{1}{2}+\frac{4}{4}+\frac{9}{8}+\frac{16}{16}+\frac{25}{32}+\frac{36}{64}+\cdots$
2. $\frac{1}{4}+\frac{2}{16}+\frac{3}{64}+\frac{4}{256}+\frac{5}{1024}+\cdots$
3. $\sum_{n=0}^{\infty}\frac{2}{3^n}$
4. $\sum_{n=1}^{\infty}\frac{1}{n^2}$
5. $\sum_{n=0}^{\infty}\frac{n!}{(2n)!}$
6. $\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}$
7. $\sum_{n=1}^{\infty}\frac{n+1}{n^3}$
8. $\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}$
9. $\sum_{n=1}^{\infty}n\left(\frac{4}{5}\right)^n$
10. $\sum_{n=1}^{\infty}\frac{(n+1)!}{n3^n}$
11. $\sum_{n=1}^{\infty}\frac{4^n}{(2n-1)!}$
12. $\sum_{n=1}^{\infty}\frac{n^n}{n!}$

In Problems 13-17, use any convergence / divergence tests you know to determine whether the given series converges.

13. $\frac{2}{6}+\frac{3}{7}+\frac{4}{8}+\frac{5}{9}+\cdots$
14. $\frac{1}{1}+\frac{3}{1}+\frac{9}{2}+\frac{27}{6}+\frac{81}{24}+\cdots$
15. $36-12+4-\frac{4}{3}+\frac{4}{9}+\cdots$
16. $\sum_{n=1}^{\infty}\frac{1}{n}$
17. $\sum_{n=1}^{\infty}\frac{n!}{2n^5}$
18. Given that $a_n>0$ for $n\geq 0$ and $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{1}{4}$, determine whether the following series converge.

    a. $\sum_{n=0}^{\infty}a_n$

    b. $\sum_{n=0}^{\infty}\frac{1}{a_n}$

    c. $\sum_{n=1}^{\infty}na_n$

    d. $\sum_{n=1}^{\infty}n^3a_n$

    e. $\sum_{n=1}^{\infty}\frac{a_n}{n}$

    f. $\sum_{n=1}^{\infty}\left(a_n\right)^2$

    g. $\sum_{n=1}^{\infty}2^n a_n$

    h. $\sum_{n=1}^{\infty}5^n a_n$

19. In Section 1, Problems 32 and 33, you found the sum of series of the form $\sum_{n=1}^{\infty}\frac{n}{r^n}$. Technically, you only showed that *if* the series converged, then they had the sums that you found. Now we can do better. Prove that $\sum_{n=1}^{\infty}\frac{n}{r^n}$ converges if $r > 1$.
20. Identify which of the following series is a power series. For those that are power series, state the center.

    a. $(x+2)+3(x+2)^2-(x+2)^3+\cdots$

    b. $2+(x-1)+(x-2)^2+(x-3)^3+\cdots$

    c. $\sum_{n=1}^{\infty}(x-3)^n$

    d. $\sum_{n=0}^{\infty}\left(x^2+4\right)^n$

    e. $\sum_{n=1}^{\infty}\frac{2^n(x+1)^{2n}}{n^3}$

    f. $\sum_{n=0}^{\infty}\tan^n x$

21. A power series $\sum_{n=0}^{\infty} c_n x^n$ converges at $x=-2$ and diverges at $x = 5$.
    a. What is the smallest possible radius of convergence of this series?
    b. What is the largest possible radius of convergence of this series?
    c. If it can be determined, state whether the series converges or diverges at the following x-values: -8, -5, -1, 0, 1, 2, 4

22. A power series $\sum_{n=1}^{\infty} c_n (x-3)^n$ converges at $x=0$ and diverges at $x=-2$.
    a. What is the smallest possible radius of convergence of this series?
    b. What is the largest possible radius of convergence of this series?
    c. If it can be determined, state whether the series converges or diverges at the following x-values: -3, -1, 2, 3, 5, 6, 8, 9

23. A power series $\sum_{n=0}^{\infty} c_n (x+1)^n$ converges at $x=5$. Of the following intervals, which could be intervals of convergence for the series?
    a. $[-5,5]$
    b. $(-3,5)$
    c. $[-8,6)$
    d. $(-7,5]$

24. A power series $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence $R = 5$. What is the radius of convergence of the power series $\sum_{n=1}^{\infty} n c_n x^{n-1}$?

25. A series $\sum_{n=0}^{\infty} f_n(x)$ converges at $x=5$ and $x=8$, but diverges at $x=6$. Can this series be a power series? Explain.

In Problems 26-40, find the radius of convergence of the given power series.

26. $1+\frac{x}{5}+\frac{x^2}{25}+\frac{x^3}{125}+\cdots$

27. $\frac{x-2}{1\cdot 2}+\frac{(x-2)^2}{2\cdot 4}+\frac{(x-2)^3}{3\cdot 8}+\frac{(x-2)^4}{4\cdot 16}+\cdots$

28. $\sum_{n=0}^{\infty} (4x)^n$

29. $\sum_{n=1}^{\infty} \frac{(x+4)^n}{n\cdot 3^n}$

30. $\sum_{n=1}^{\infty} \frac{3^n}{n^2}(x+1)^n$

31. $\sum_{n=1}^{\infty} \frac{n!x^n}{n^3 4^n}$

32. $\sum_{n=0}^{\infty} \frac{n+1}{n^2}(x-5)^n$

33. $\sum_{n=0}^{\infty} \frac{(x+5)^n}{3^n}$

34. $\sum_{n=0}^{\infty} \frac{2^n}{n!}(x-1)^n$

35. $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^n}{n^2}$

36. $\sum_{n=0}^{\infty} \frac{\cos(n\pi)\cdot(x+2)^n}{3^n}$

37. $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{(x-4)^{2n}}{4n}$

38. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

39. $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$

40. $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$

41. A function $f(x)$ has a power series with radius of convergence $R = 15$. What is the radius of convergence of the power series for $f(5x)$?

42. Suppose that a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges with radius of convergence R. Show that $\sum_{n=0}^{\infty} nc_n (x-a)^n$ also has radius of convergence R.

43. Another convergence test that can be useful for working with power series (or series of constants) is the **root test**. In this test, we examine $\lim_{n\to\infty} \sqrt[n]{a_n}$, again assuming that $a_n > 0$. As with the ratio test, Σa_n converges if this limit is less than 1 and diverges if this limit is greater than 1. No conclusion can be drawn about Σa_n if the limit equals 1.

    Use the root test to determine whether the following series converge. If the test is inconclusive, state that as well.

    a. $\sum_{n=1}^{\infty} \frac{n}{3^n}$

    b. $\sum_{n=1}^{\infty} \frac{2}{n^3}$

    c. $\sum_{n=1}^{\infty} \left(\frac{2n+1}{n}\right)^n$

    d. $\sum_{n=0}^{\infty} \frac{1}{n^n}$

44. Use any convergence / divergence tests you know to determine whether the following series converge.

    a. $\sum_{n=1}^{\infty} \left(\frac{3}{n} - \frac{3}{n+1}\right)$

    b. $\sum_{n=0}^{\infty} \frac{2^n}{(3n)!}$

    c. $\sum_{n=1}^{\infty} \frac{1}{e^n}$

    d. $\sum_{n=0}^{\infty} \frac{3^n+1}{3^n}$

    e. $\sum_{n=0}^{\infty} \frac{3^n}{4^n}$

    f. $\sum_{n=1}^{\infty} \frac{2^n}{n}$

45. There are several refinements of the ratio test that are less frequently inconclusive. One of these is known as the **Raabe test**. In this test, we examine $\lim_{n\to\infty} \left[n\left(\frac{a_n}{a_{n+1}} - 1\right)\right]$.

    Unlike with the ratio and root tests, Σa_n converges if this limits is *greater than* 1 and diverges if this limit is less than 1. (This makes sense; the fraction is upside down relative to the ratio test.) Again, no conclusion can be drawn if the limit equals 1. Even this refinement has inconclusive cases.

    Use the Raabe test to determine whether the following series converge. If the test is inconclusive, state that.

    a. $1 + \frac{1}{2} + \frac{1\cdot 3}{2\cdot 4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} + \cdots + \frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot\cdots\cdot(2n)} + \cdots$ *

    b. $\frac{2}{5} + \frac{2\cdot 4}{5\cdot 7} + \frac{2\cdot 4\cdot 6}{5\cdot 7\cdot 9} + \cdots + \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{5\cdot 7\cdot 9\cdot\cdots\cdot(2n+3)} + \cdots$

    c. $\frac{2}{4} + \frac{2\cdot 5}{4\cdot 7} + \frac{2\cdot 5\cdot 8}{4\cdot 7\cdot 10} + \cdots + \frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)}{4\cdot 7\cdot 10\cdot\cdots\cdot(3n+1)} + \cdots$

    d. $\sum_{n=1}^{\infty} \left[\frac{2\cdot 4\cdot\cdots\cdot(2n)}{5\cdot 7\cdot\cdots\cdot(2n+3)}\right]^{2/3}$

46. Not all series that look like those in Problem 45 require the Raabe test. Use the ratio test to determine whether the series $\frac{1}{2} + \frac{1\cdot 3}{2\cdot 5} + \frac{1\cdot 3\cdot 5}{2\cdot 5\cdot 8} + \cdots + \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-1)} + \cdots$ converges.

47. As mentioned in a footnote in this section, the function $h(x) = \sqrt{x}$ can be represented as a power series centered at $x = 4$ as
$$2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n-3)!!}{n!\cdot 2^{3n-1}} (x-4)^n$$
where !! is the double factorial. By definition, $k!! = k\cdot(k-2)!!$ and $0!! = (-1)!! = 1$.

    a. Use the ratio test to determine the radius of convergence of this power series.

    b. Plugging in 0 for x gives the series
    $$2 - \sum_{n=1}^{\infty} \frac{(2n-3)!!\cdot(4)^n}{n!\cdot 2^{3n-1}}.$$
    Use the Raabe test to determine whether this series converges.

    c. Plugging in 8 for x gives the series
    $$2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n-3)!!\cdot 4^n}{n!\cdot 2^{3n-1}}.$$
    Ignore the alternating factor $(-1)^{n+1}$ for now (we will justify doing this in

* If we use the notation of double factorials, then the general term of this series is simply $\frac{(2n-1)!!}{(2n)!!}$.

Section 8), and determine whether this series converges.

d. Give the complete interval of convergence for this power series.

Not all series representations of functions are power series. However, many non-power series are also interesting. In Problems 48-50, you will explore a few.

48. Consider the series $\sum_{n=0}^{\infty}(\sin x)^n$. Find all values of x for which this series converges. Does the series converge on an interval like power series do? Graph a few partial sums of this series.

49. Consider the geometric series $\sum_{n=0}^{\infty}\left(\frac{x^2-13}{12}\right)^n$.

a. Why is this not a power series?

b. To what function does this series converge?

c. For what values of x does the series converge?

d. Graph a few partial sums of this series along with your answer to part (b).

50. As we know, the function $f(x)=\frac{1}{1-x}$ can be represented as the power series $\sum_{n=0}^{\infty}x^n$. There are other series representations for this function, though.

a. Show that $f(x)=\frac{-\frac{1}{x}}{1-\frac{1}{x}}$ for $x\neq 0$.

b. Expand the form of $f(x)$ from part (a) as a geometric series. Why is this series not a power series? (This is an example of something called a Laurent series.)

c. Graph some partial sums of your series from part (b) along with partial sums of the same degree for $\sum_{n=0}^{\infty}x^n$. You may also want to include the graph of *f*.

51. Show that the power series $f(x)=\sum_{n=3}^{\infty}\frac{2x^n}{n!}$ solves the differential equation $y'=x^2+y$.

Section 7 – Positive-Term Series

Our goal in this chapter is to work with power series, but that requires our being able to determine the values of x for which a series converges—the interval of convergence. As we saw in Section 6, applying the ratio test to the general term of a series was useful in determining how wide the interval of convergence is. However, the ratio test is always inconclusive at the endpoints of the interval. In this section and the next, we will develop tools for examining the endpoints and answering the question of convergence there. With these tools, we will be able to determine a complete interval of convergence for many power series, including the Taylor series that we commonly encounter. In this section we will deal exclusively with series whose terms are always positive (or at least non-negative). Series whose terms have varying signs will be examined in Section 8.

The Integral Test

Let us begin by examining two series: $H=\sum_{n=1}^{\infty}\frac{1}{n}$ and $S=\sum_{n=1}^{\infty}\frac{1}{n^2}$. In Section 1 we saw that H diverges and it was stated that S converges (to $\pi^2/6$), but another look at both will be informative. We'll start with H.

$H=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$. Graphically, we can think of this sum as adding the areas of an infinite number of rectangles. Each rectangle will have width 1, and the n^{th} rectangle will have height $1/n$. (See Figure 7.1.) But with the graph of $y=\frac{1}{x}$ superimposed over these rectangles, we see that this sum is nothing more than a left Riemann sum for the improper integral $\int_1^{\infty}\frac{1}{x}\,dx$. Furthermore, the picture certainly suggests that $\sum_{n=1}^{\infty}\frac{1}{n}>\int_1^{\infty}\frac{1}{x}\,dx$. We know from our study of improper integrals that $\int_1^{\infty}\frac{1}{x}\,dx$ diverges. Hence, the harmonic series H must also diverge. If the area under the curve $y=\frac{1}{x}$ is unbounded, then the series, which represents *more* area, must be unbounded as well.

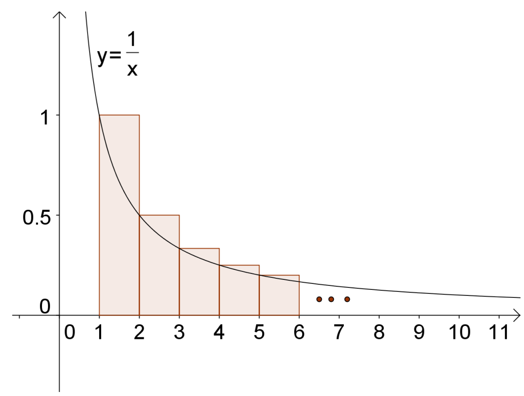

Figure 7.1: *H* as a left Riemann sum

Now let's turn to S. If we draw the same kind of picture, with rectangles of width 1 and height $1/n^2$, it is actually not very helpful (Figure 7.2). We know that $\int_1^{\infty}\frac{1}{x^2}\,dx$ converges, and we see from the picture that $\sum_{n=1}^{\infty}\frac{1}{n^2}>\int_1^{\infty}\frac{1}{x^2}\,dx$. But what does that mean? *How much* bigger than $\int_1^{\infty}\frac{1}{x^2}\,dx$ is $\sum_{n=1}^{\infty}\frac{1}{n^2}$? If the sum is only a little bigger than the integral, then it is reasonable to guess that S converges. But if S is much bigger than the integral, the series may diverge even though the integral converges. We simply cannot conclude anything from this picture. However, we can flip it around and look at a right Riemann sum for $\int_1^{\infty}\frac{1}{x^2}\,dx$. Figure 7.3 shows that we can do this, though we end up losing the first, tallest rectangle in the process. (If we kept that rectangle and shifted it over, we would be looking at a Riemann sum for $\int_0^\infty \frac{1}{x^2}\,dx$. This introduces a new kind of improper-ness to the integral: unbounded behavior of the integrand for x-values near zero. Since we only care about long-term behavior and have no need to be entangled in what is going on for x-values near zero, we sacrifice the first rectangle and focus on what is left over. Besides, this rectangle only represents the first term of the infinite series, the loss of which will not affect its convergence.) We see from Figure 7.3 that $\sum_{n=2}^{\infty}\frac{1}{n^2} < \int_1^\infty \frac{1}{x^2}\,dx$. Hence, since $\int_1^\infty \frac{1}{x^2}\,dx$ converges, $\sum_{n=2}^{\infty}\frac{1}{n^2}$ must converge as well. Geometrically, if the area bounded by $y=\frac{1}{x^2}$ from $x=1$ to ∞ is finite, then the area represented by $\sum_{n=2}^{\infty}\frac{1}{n^2}$ must be as well. Since $\sum_{n=2}^{\infty}\frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty}\frac{1}{n^2}$ also converges.

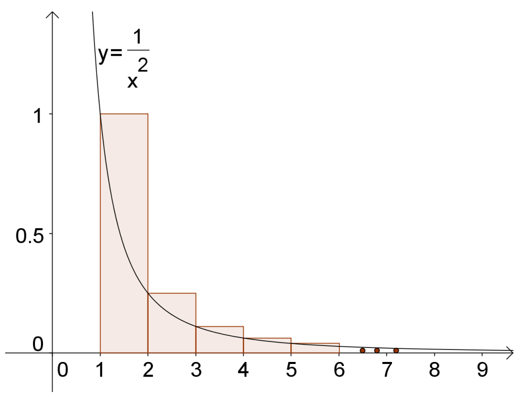

Figure 7.2: ***S*** **as a left Riemann sum**

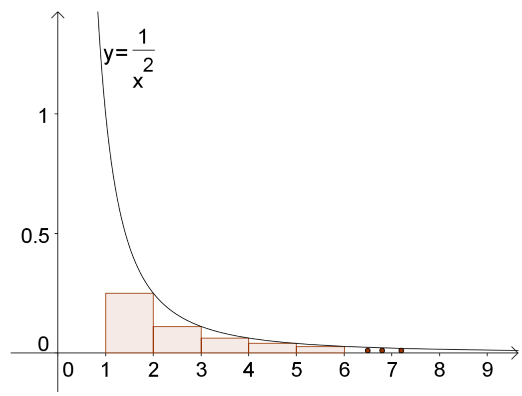

Figure 7.3: ***S*** **as a right Riemann sum**

I would like to make two remarks about the preceding arguments. First, notice how important it was that the terms of the series being discussed were positive (or at least non-negative). If the terms of $\sum_{n=2}^{\infty}\frac{1}{n^2}$ were sometimes negative, then the fact that the Riemann sum corresponding to $\sum_{n=2}^{\infty}\frac{1}{n^2}$ is less than $\int_1^\infty \frac{1}{x^2}\,dx$ would be meaningless; the series could still diverge if its terms were negative and huge. But because the terms of $\sum_{n=2}^{\infty}\frac{1}{n^2}$ are actually positive, they are trapped between 0 and the values of $\frac{1}{x^2}$. This is what allows us to conclude something meaningful.

Second, we should look at what happens if we switch the Riemann sum in Figure 7.1 to a right Riemann sum. This is shown in Figure 7.4, and again we lose the first rectangle. Figure 7.1 tells us that $\sum_{n=2}^{\infty}\frac{1}{n} < \int_1^\infty \frac{1}{x}\,dx$. But again the question is: How much? Is the area representing the sum small enough that it converges even when the area represented by the improper integral diverges? The picture cannot answer this question for us. The moral is that being smaller than something divergent is not enough to ensure convergence, just as being bigger than something convergent was not

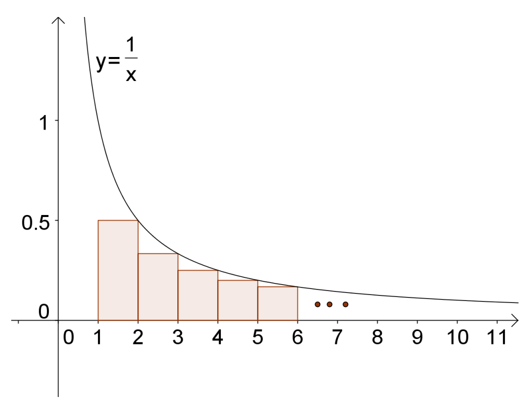

Figure 7.4: ***H*** **as a right Riemann sum**

enough to guarantee divergence in the case of $\sum_{n=1}^{\infty}\frac{1}{n^2}$. We will see this idea again.

The examples of H and S suggest that we can infer the behavior of a particular series by looking at a corresponding improper integral. This is stated precisely by the integral test.

Theorem 7.1 – The Integral Test
If $f(x)$ is a positive, continuous, and decreasing function such that $f(n) = a_n$ for all n at least as large as some threshold value N, then $\sum_{n=N}^{\infty} a_n$ and $\int_N^{\infty} f(x)dx$ either both converge or both diverge. In other words, as the improper integral behaves, so does the series, and vice versa.

The stuff about the threshold value N sounds complicated, but it is actually there to make things simpler. First, it gives us the freedom to not specify a starting index value in the theorem, which lets us be a bit more general. Second, if the function f doesn't satisfy all the hypotheses of the theorem (being positive, continuous, and decreasing) *until* some x-value, that's fine. As long as $f(x)$ matches the values of the series terms and *eventually* acts as needed, the conclusion of the theorem follows. Remember that convergence is not affected by the first "few" terms of a series, only by its long-run behavior. We do not mind if f behaves erratically before settling down at $x = N$.

In addition to the requirement that $f(n) = a_n$, there are quite a few requirements on the function f in the integral test. The technical reason for these requirements is that they are necessary to prove the theorem. I am not going to provide a proof, but perhaps we can understand conceptually what these requirements do for us. (In the meantime, if there is a proof in your main text, read it and look for where each of the hypotheses is used.) We are interested in the series Σa_n, but we are using f as a proxy for $\{a_n\}$. This is a little awkward since $\{a_n\}$ is only defined for integer values of n while f is presumably defined for many more x-values than just the integers. The hypotheses of the integral test are there to make sure that f continues to act like we expect it to even between the integer values of x. Look back at the figures that we used in discussing H and S; they show that in each case the function f and the sequence $\{a_n\}$ only agree at the corners of the rectangles. If f were doing wild things between these points—crazy oscillation, unexpected growth, or vertical asymptotes—it would no longer be reasonable to expect the convergence behavior of the integral of $\int_N^{\infty} f(x)dx$ to match that of $\sum_{n=N}^{\infty} a_n$. The hypotheses of the integral test, while fussy, are necessary to ensure that the improper integral will in fact act as a reasonable substitute for the series.

Example 1

Determine whether the series $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ converges.

Solution

Obviously, we are going to use the integral test since that is what this section is about. But let's pause and think about *why*. For one thing, the series passes the n^{th} term test: $\lim_{n\to\infty}\frac{n}{n^2+1} = 0$. This means that convergence is a possibility. The series is not geometric, and the ratio test is inconclusive (try it), so we are really left with no other alternatives. Additionally, if we translate the terms $a_n = \frac{n}{n^2+1}$ into the function $f(x)=\dfrac{x}{x^2+1}$, we see something that we can antidifferentiate. This suggests the integral test is worth trying.

Begin by noting that for $x \ge 1$ $f(x)$ is positive, decreasing, and continuous. (In most cases, a graph is sufficient to convince yourself of these things. Continuity and positivity for $x > 0$ should be obvious. If you want to prove that the function decreases, examine the derivative.) Now we try to evaluate $\int_1^\infty \dfrac{x}{x^2+1}dx$.

$$\begin{aligned}\int_1^\infty \frac{x}{x^2+1}dx &= \lim_{b\to\infty}\int_1^b \frac{x}{x^2+1}dx \\ &= \lim_{b\to\infty}\left[\tfrac{1}{2}\ln\left(x^2+1\right)\right]_0^b \\ &= \tfrac{1}{2}\lim_{b\to\infty}\ln\left(b^2+1\right)-\tfrac{1}{2}\ln(1)\end{aligned}$$

However, this limit does not exist; the improper integral diverges. By the integral test, we conclude that the series $\sum_{n=1}^\infty \dfrac{n}{n^2+1}$ diverges as well. ◊

Example 2

Determine whether the series $\sum_{n=0}^\infty \dfrac{1}{n^2+1}$ converges.

Solution

Again, we will use the integral test, noting that for $x \ge 0$ $f(x)=\dfrac{1}{x^2+1}$ is positive, decreasing, and continuous.

$$\begin{aligned}\int_0^\infty \frac{1}{x^2+1}dx &= \lim_{b\to\infty}\int_0^b \frac{1}{x^2+1}dx \\ &= \lim_{b\to\infty}\left[\tan^{-1}(x)\right]_0^b \\ &= \lim_{b\to\infty}\tan^{-1}(b)-\tan(0) \\ &= \pi/2\end{aligned}$$

This time the improper integral converges. We conclude that $\sum_{n=0}^\infty \dfrac{1}{n^2+1}$ converges as well. ◊

<u>IMPORTANT</u>: The *fact* that the improper integral in Example 2 converges tells us something useful about the corresponding series. The *value* to which the improper integral converges is all but irrelevant. Though the value of the improper integral in Example 2 was $\pi/2$, it is <u>**not**</u> the case that this is the value of the series $\sum_{n=0}^\infty \dfrac{1}{n^2+1}$. Indeed, $s_2 = 1.7$, which already exceeds $\pi/2$, and further terms will only increase the values of the partial sums. Look again at Figures 7.1-7.4. Clearly the values of the Riemann sums (i.e., the series) were not equal to the areas bounded by the curves (i.e., the improper integrals). It is a common and understandable mistake to assume that when the integral test shows convergence, the value of the series is the same as the value of the improper integral. I think this comes from a basic dissatisfaction we have with the fact that we can find exact values of series so rarely. But do not fall for the trap of conflating the two values.

Practice 1

Determine whether the series $\sum_{n=2}^{\infty}\frac{n^2}{n^3-4}$ converges.

Estimates Based on Integrals (Optional)

Despite the comments following Example 2 warning you not to confuse the value of a convergent improper integral with a corresponding infinite series, it happens that the value of the improper integral is not completely useless to us. Figure 7.2 suggests that $\sum_{n=1}^{\infty}\frac{1}{n^2} > \int_1^{\infty}\frac{1}{x^2}dx = 1$. On the other hand, Figure 7.3 shows that $\sum_{n=2}^{\infty}\frac{1}{n^2} < \int_1^{\infty}\frac{1}{x^2}dx$, or equivalently

$$1+\sum_{n=2}^{\infty}\frac{1}{n^2} < 1+\int_1^{\infty}\frac{1}{x^2}dx$$
$$\sum_{n=1}^{\infty}\frac{1}{n^2} < 2.$$

Indeed, experimenting with partial sums should verify for you that $1 < \sum_{n=1}^{\infty}\frac{1}{n^2} < 2$.

In fact, you can quickly find from your calculator that to six decimal places $s_{100} = 1.634984$. The tail, the unused part of the series, is $\sum_{n=101}^{\infty}\frac{1}{n^2}$. We can use the same reasoning from the previous paragraph to get a sense for the size of the tail. On the one hand, $\sum_{n=101}^{\infty}\frac{1}{n^2}$ must be larger than $\int_{101}^{\infty}\frac{dx}{x^2}$. This is strictly in analogy with the discussion above, but if you are not convinced, draw out a left Riemann sum for $\int_{101}^{\infty}\frac{dx}{x^2}$. Similarly, a right Riemann sum implies that $\sum_{n=102}^{\infty}\frac{1}{n^2} < \int_{101}^{\infty}\frac{dx}{x^2}$ or $\sum_{n=101}^{\infty}\frac{1}{n^2} < \frac{1}{101^2}+\int_{101}^{\infty}\frac{dx}{x^2}$. Putting it all together, we find that

$$\int_{101}^{\infty}\frac{dx}{x^2} < \sum_{n=101}^{\infty}\frac{1}{n^2} < \frac{1}{101^2}+\int_{101}^{\infty}\frac{dx}{x^2}$$
$$\frac{1}{101} < \sum_{n=101}^{\infty}\frac{1}{n^2} < \frac{1}{101^2}+\frac{1}{101}.$$

Now we know that $s_{100} = 1.634984$ and that the tail has a value somewhere between 0.009901 and 0.010097. Thus the actual value of the series, s_{100} + tail, is somewhere between 1.644885 and 1.654884. The actual value of $\pi^2/6$ is indeed within this range. This approach should allow you to find the value of this series to any degree of precision that you would like. You are only limited by the number of decimal places that your calculator displays.

The sort of reasoning we have been using applies to any convergent series whose terms are decreasing, not just to ones for which we use the integral test to determine convergence. For such series we have

$$\int_N^\infty f(x)dx < \sum_{n=N}^\infty a_n < a_N + \int_N^\infty f(x)dx. \quad (7.1)$$

We can always compute a partial sum simply by adding up a finite number of terms. The strength of (7.1) is that it gives us a way to estimate the size of the tail as well. Even if the improper integrals in (7.1) cannot be evaluated analytically, a numerical approximation of them is still useful for approximating the value of a series.

p-Series

After all this, I must admit that I try to avoid the integral test at all costs. The hypotheses can be a chore to verify, and the integration can also require quite a bit of effort. While some series might scream for the integral test, perhaps something like $\sum_{n=1}^\infty ne^{-n^2}$ is an example, in general the integral test is a test of last resort. Most series can be handled using simpler tests.

Why, then, did we spend so much time on the integral test? The integral test gives us a very simple and extremely useful convergence test as a consequence. First, though, we need a smidge of vocabulary.

Definition: A ***p*-series** is a series of the form $\sum_{n=1}^\infty \frac{1}{n^p}$ where p is a number.

We have already encountered p-series a few times. Both H and S are p-series. In H, the value of p is 1, while in S it is 2. p need not be a whole number; $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ is a p-series with $p = 1/2$. While p can be negative, we are typically interested in positive p-values. If p is negative or zero, then the series will diverge in a fairly straight-forward manner. (Why?) Things that look *like* p-series generally are not p-series. So while it is tempting to say that $\sum_{n=1}^\infty \frac{1}{n^2+1}$ is a p-series with $p = 2$ since it looks so much like the form in the definition, that would be wrong. On the other hand, we will view a series like $\sum_{n=1}^\infty \frac{3}{n^5}$ as being essentially a p-series with $p = 5$. Theorem 1.1 tells us that this series is equivalent to $3 \cdot \sum_{n=1}^\infty \frac{1}{n^5}$, and this latter form is clearly a p-series times some constant multiple.

The convergence of any p-series can be determined by using the integral test. $f(x) = \frac{1}{x^p}$ is positive and continuous for $x > 0$. And if p is positive, f is decreasing for $x > 0$. Thus we will be interested in computing $\int_1^\infty \frac{1}{x^p}dx$. But as we know from our study of improper integrals, this integral converges if and only if $p > 1$. This means that there is no real reason to actually evaluate the integral; we already know everything we need, and the theorem follows immediately.

Theorem 7.2 – The *p*-series Test
A p-series $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if and only if $p > 1$.

This is a pretty simple test. Only the geometric series test comes close to its simplicity.

Example 3
Determine whether the following series converge.

a. $\sum_{n=8}^{\infty} \frac{1}{\sqrt{n^3}}$ b. $\sum_{n=1}^{\infty} \frac{1}{n^{0.1}}$

Solution

a. First of all, don't let the starting value of the index scare you. Remember that convergence does not depend on the first few terms, only on the long-run behavior of the terms. In any event, rewrite $\frac{1}{\sqrt{n^3}}$ as $\frac{1}{n^{3/2}}$ to see that this is a *p*-series with $p = 3/2$. Since $3/2 > 1$, the series converges.

b. This series is a *p*-series with $p = 0.1 < 1$. This series diverges by the *p*-series test. ◊

Practice 2
Determine whether the following series converge.

a. $\sum_{n=1}^{\infty} \frac{1}{n^8}$ b. $\sum_{n=1}^{\infty} \frac{\sqrt[4]{n}}{n}$ c. $\sum_{n=1}^{\infty} n^{-1/3}$

Comparison Tests

Between the geometric series test and the *p*-series test, we can quickly and effectively test a host of comparatively simple series for convergence. Of even more value, though, is the fact that these types of series can be used as a basis of comparison for more complicated series. We formalize this idea in our final two positive-term convergence tests.*

Theorem 7.3 – The Direct Comparison Test
Suppose $a_n \geq 0$ and $b_n \geq 0$ for all *n* (or at least all *n* past some threshold *N*). Further suppose that $a_n \leq b_n$ for all *n* (or, again, all *n* past *N*).

If $\sum_{n=N}^{\infty} b_n$ converges, then $\sum_{n=N}^{\infty} a_n$ converges.

If $\sum_{n=N}^{\infty} a_n$ diverges, then $\sum_{n=N}^{\infty} b_n$ diverges.

If either $\sum_{n=N}^{\infty} a_n$ converges or $\sum_{n=N}^{\infty} b_n$ diverges, then no conclusion can be drawn about the other series based on this test.

Here's the analogy that makes sense of this theorem. Suppose we have two calculus students, Anna and Brian. Anna is on the short side, while Brian is quite tall. They are about to walk through a door, but it's a strange door; we don't know how tall the doorway is, so we don't know whether either Anna or Brian will fit through. Brian goes first and we find that Brian does in fact fit through the door. This means that Anna must also fit; she is shorter than Brian, and he got through. But suppose Brian had not fit through the door. In that case, we would not have been able to predict whether Anna would make it or not. It may be that the doorway is *really* short, and that is why Brian could not make it. Or it could be that he only just barely failed to pass through, and Anna would have made it had she tried. Now flip the thought experiment around. If Anna passes through the door, what does that tell us about Brian's chances? Nothing. The doorway might be just tall enough to permit Anna while still blocking Brian, or it could be

* There are many more, but the ones in this chapter are plenty for a first calculus course.

that the door is immense and Brian will fit through as well. But if Anna cannot get through the door, then Brian certainly has no hope.

In this analogy, Anna is playing the role of a generic term a_n while Brian stands for the generic b_n. Anna's being shorter than Brian corresponds to the hypothesis that $a_n \le b_n$. "Fitting through the door" means that the series converges. If Anna fits, then that means that $\sum_{n=N}^{\infty} a_n$ converges. Failure to get through the door represents divergence. The basic idea is that if a series converges, then a "smaller series" will also converge, though this is not a very precise way of saying it. Similarly, if a series diverges, then a "bigger series" will also diverge. But be careful to recognize the inconclusive cases. If a series diverges, then a "smaller series" may converge or diverge; more information is needed.

Hopefully, the idea behind the direct comparison test is relatively clear. Actually applying it is a bit of work. The two key pieces are (1) finding a simple series to compare against whose convergence we know, and (2) actually making the direct comparison. Very frequently our comparisons will be against either p-series or geometric series.

Example 4

Determine whether the following series converge.

a. $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$ b. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ c. $\sum_{n=0}^{\infty} \frac{1}{3^n-1}$

Solution

a. This series looks a lot like $\sum_{n=1}^{\infty} \frac{1}{n^2}$. When n is very large, the 5 will hardly make much difference. So we will compare $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$ to $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (The broader lesson here is to focus on the dominant terms within a_n to figure out what your comparison series will be.) Note that since $n^2+5 > n^2$ for all n, $\frac{1}{n^2+5} < \frac{1}{n^2}$. Since we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (it is a p-series with $p = 2 > 1$), it follows by the direct comparison test that $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$ converges.

b. For this series it is not as obvious what series to choose for comparison. But since we want something simple, we might try a p-series. There are no powers in the series we are exploring, so let's try the trivial power of 1 and compare to $\sum_{n=2}^{\infty} \frac{1}{n}$. For all $n \ge 2$, $n > \ln n$. Thus, for $n \ge 2$, $\frac{1}{n} < \frac{1}{\ln n}$. Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (it is the harmonic series), by direct comparison $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ must diverge as well.

c. Focusing on dominant terms leads us to ignore the 1 in the denominator and think about $\sum_{n=0}^{\infty} \frac{1}{3^n}$. Unfortunately, the comparison is all wrong here. $3^n > 3^n - 1$ for all n, so $\frac{1}{3^n} < \frac{1}{3^n-1}$. We know that $\sum_{n=0}^{\infty} \frac{1}{3^n}$ converges since it is geometric with $|r| = \frac{1}{3} < 1$. However, we are now comparing a "bigger series" to a convergent series. The direct comparison test is inconclusive in this case. You can hunt for other series to make a better comparison; I am sure it can be done. But for now we will pass over this series and come back to it later with a slightly different tool. ◊

Example 5

Use direct comparison to show that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

Solution

Virtually every textbook has this example, so I feel obliged to include it as well, even if the convergence of this series is best determined by the ratio test. (Most books introduce the comparison tests long before the ratio test.) Notice that for $n \ge 4$, $n! > 2^n$. As a consequence, $\frac{1}{n!} < \frac{1}{2^n}$ for $n \ge 4$. Now we can compare. $\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges (it is geometric with $|r| = \frac{1}{2} < 1$). Therefore, by direct comparison, $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges as well.

We can go a bit further. From the comparison we are using, we can show that $\sum_{n=4}^{\infty} \frac{1}{n!} < \sum_{n=4}^{\infty} \frac{1}{2^n}$. The latter series is geometric, so we can actually find its sum. I leave the details to you, but the sum is 1/8.

$$\sum_{n=4}^{\infty} \frac{1}{n!} < \frac{1}{8}$$

$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \sum_{n=4}^{\infty} \frac{1}{n!} < \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{8}$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} < 2.792$$

We actually get a useful result from this comparison: an upper bound on the value of the series in question. This bound will turn out to be important later. ◊

The tricky part of using a comparison test, as you have probably guessed, is in finding the series to compare against. Focusing on dominant terms is always a good start, but is not always the key, as Examples 4b and 5 show. The important thing is to try something and not be afraid to fail. You will probably pick the wrong thing to compare to every now and again. That's okay. Pick yourself up and try again.

Practice 3

Determine whether the following series converge.

a. $\sum_{n=0}^{\infty} \frac{5^n + 1}{2^n - 1}$ b. $\sum_{n=6}^{\infty} \frac{n-5}{2^n + 1}$

Let's return to Example 4c. We failed to determine the convergence of $\sum_{n=0}^{\infty} \frac{1}{3^n - 1}$ by direct comparison. If your gut tells you that $\sum_{n=0}^{\infty} \frac{1}{3^n - 1}$ should converge because its terms are so much like those of $\sum_{n=0}^{\infty} \frac{1}{3^n}$ when *n* is large, you've got a smart gut. We formulate "being so much like" in the following theorem.

Theorem 7.4 – The Limit Comparison Test
Suppose $a_n > 0$ and $b_n > 0$ for all n (or at least all n past a certain threshold N).

If $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ exists and is both positive and finite, then $\sum\limits_{n=N}^{\infty} a_n$ and $\sum\limits_{n=N}^{\infty} b_n$ either both converge or both diverge.

Let's think for a moment about what the limit comparison test says. Suppose the series Σa_n happens to converge. This means that the terms of the series are dwindling to zero fast enough that the partial sums can approach a finite number. Now enter another series Σb_n. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ is both positive and finite, then the terms of the two series have, if not the same decay rates, then at least rates that are comparable. Their decay rates are related by at worst a constant multiple, and that puts them at least in the same order of magnitude. So if the terms of Σa_n go to zero quickly enough for convergence, the terms of Σb_n must as well. And the converse is true for divergence.

The trick, as always, is to decide what known series to compare against.

Example 6
Determine whether the following series converge.

a. $\sum\limits_{n=0}^{\infty} \frac{1}{3^n - 1}$ b. $\sum\limits_{n=1}^{\infty} \frac{2n^2 + 2n - 1}{n^{5/2} - 3n + 8}$

Solution

a. We compare, as we always wanted to, to the convergent geometric series $\sum\limits_{n=0}^{\infty} \frac{1}{3^n}$. We must evaluate the limit of a_n / b_n. It does not really matter which series plays the role of Σa_n and which plays the role of Σb_n.

$$\lim_{n\to\infty} \frac{\frac{1}{3^n - 1}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{3^n}{3^n - 1} = 1$$

If you must, you can use l'Hôpital's Rule in evaluating the limit. The point is that the limit is positive and finite. Thus, since $\sum\limits_{n=0}^{\infty} \frac{1}{3^n}$ converges, $\sum\limits_{n=0}^{\infty} \frac{1}{3^n - 1}$ converges as well by the limit comparison test.

b. If we focus on dominant terms, we are led to compare to the series $\sum\limits_{n=1}^{\infty} \frac{n^2}{n^{5/2}} = \sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. (I am also ignoring the coefficient of 2 since this can be factored out of the series; it certainly won't affect the issue of convergence.) The helper series diverges because it is a p-series with $p = \frac{1}{2} \le 1$. Now for the limit.

$$\lim_{n\to\infty} \frac{1/\sqrt{n}}{\left(2n^2 - 2n - 1\right)/\left(n^{5/2} - 3n + 8\right)} = \lim_{n\to\infty} \frac{n^{5/2} - 3n + 8}{\sqrt{n}\left(2n^2 - 2n + 1\right)} = \lim_{n\to\infty} \frac{n^{5/2} - 3n + 8}{2n^{5/2} - 2n^{3/2} + \sqrt{n}} = \frac{1}{2}$$

Again, the limit is positive and finite, so the two series under consideration have the same convergence behavior. By the limit comparison test, we conclude that $\sum\limits_{n=1}^{\infty} \frac{2n^2 + 2n - 1}{n^{5/2} - 3n + 8}$ diverges.

Quick note: Had we set up the limit in part (b) the other way, with the roles of Σa_n and Σb_n swapped, we would have gotten the reciprocal in the limit, namely 2. This is still positive and finite, and we would have drawn the same conclusion. This is why it does not matter which series is Σa_n; the choice might affect the value of the limit, but not whether it is a positive, finite number. ◊

If the limit in the limit comparison test works out to be either 0 or ∞, then Σa_n and Σb_n need not have the same convergence behavior. Though there are cases in which we can draw conclusions from limits of 0 and ∞, it is often best to do a direct comparison test or a different kind of test altogether. Problems 59 through 62 give you an opportunity to see why interpreting these limits requires a bit more care.

Practice 4

Determine whether the series $\sum_{n=2}^{\infty}\frac{n^2-1}{n^3}$ converges.

At this point, we have a lot of tests that can be applied to positive-term series: the n^{th} term, geometric series, ratio, integral, p-series, direct comparison, and limit comparison tests. (If you've been doing all the problems, you've also seen the root test, the Raabe test, and a method for dealing with telescoping series. And there are more out there: a couple named after Gauss, at least one named after Cauchy… the list goes on.) All have different hypotheses that must be met, and all have different criteria for interpretation. Keeping all of these details straight can be confusing. Your primary text may have a table or chart summarizing these tests, but I think the best thing is for you to create one for yourself. Organizing this information in a way that makes sense to you is the best way to internalize it.

To be successful with series, you need to develop a sense for when to use each test. Geometric and p-series announce themselves fairly clearly, so there should never be any doubt as to when to apply those tests. The ratio test is great when dealing with factorials and exponential factors (and also power series, though never at their endpoints). However, the ratio test is terrible for p-series and p-like series. Convergence of a series can be ruled out with the n^{th} term test, but we can never show convergence with it. If the general term looks like something you can integrate, there is always the integral test, though the comparison tests should often be considered before going there. The only way to develop your intuition and skill at selecting which test to use for what series is to practice working lots of problems. And you have to make a mistake every now and again by choosing the "wrong" test or a comparison that is not helpful. When you do err, don't give up! Figure out what went wrong and give it your best second effort. That's the only way to learn this stuff.

Answers to Practice Problems

1. The corresponding function $f(x)=\frac{x^2}{x^3-4}$ is positive, decreasing and continuous for $x\geq 2$.

$$\begin{aligned}\int_2^{\infty}\frac{x^2}{x^3-4}\,dx &= \lim_{b\to\infty}\int_2^{b}\frac{x^2}{x^3-4}\,dx\\ &= \lim_{b\to\infty}\left[\tfrac{1}{3}\ln\left(x^3-4\right)\right]_2^b\\ &= \tfrac{1}{3}\lim_{b\to\infty}\ln\left(b^3-4\right)-\tfrac{1}{3}\ln 4\end{aligned}$$

As in Example 1, this limit, and hence the improper integral, diverges. By the integral test, we conclude that the series diverges.

2. All of these series are p-series.

a. $p = 8 > 1$, so the series converges.

b. $\frac{\sqrt[4]{n}}{n}=\frac{n^{1/4}}{n^1}=\frac{1}{n^{3/4}}$. $p=\frac{3}{4}\leq 1$, so this series diverges.

c. $n^{-1/3} = \frac{1}{n^{1/3}}$. $p = \frac{1}{3} \le 1$, so this series diverges.

3. a. When n is large, neither of the 1s will make much difference. We ignore them and focus on $\sum_{n=0}^{\infty} \frac{5^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{5}{2}\right)^n$. This is a divergent geometric series ($|r| = \frac{5}{2} > 1$). It is clear that $\frac{5^n}{2^n} < \frac{5^n + 1}{2^n - 1}$ because the fraction on the left side has both a smaller numerator and a larger denominator. Therefore, by direct comparison, $\sum_{n=0}^{\infty} \frac{5^n + 1}{2^n - 1}$, diverges.

b. If we focus on the dominant terms, we are left with $\sum_{n=6}^{\infty} \frac{n}{2^n}$. This is not one of our stock "simple" series (geometric or p-series). But it is simple enough that we *can* determine its convergence. The easy way is to cite Section 6, Problem 19 in which we proved that this series would converge. If you skipped that problem, then we proceed by the ratio test (since the ratio test does well with exponential terms).

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right) = \lim_{n\to\infty} \left(\frac{n+1}{n} \cdot \frac{1}{2} \right) = \frac{1}{2} < 1 .$$

Therefore, the "helper" series $\sum_{n=6}^{\infty} \frac{n}{2^n}$ converges by the ratio test. Now $\frac{n-5}{2^n+1} < \frac{n}{2^n}$ because the fraction on the right has both a larger numerator and a smaller denominator. Therefore, $\sum_{n=6}^{\infty} \frac{n-5}{2^n+1}$ converges by direct comparison to $\sum_{n=6}^{\infty} \frac{n}{2^n}$.

4. The obvious choice for comparison is $\sum_{n=2}^{\infty} \frac{n^2}{n^3} = \sum_{n=2}^{\infty} \frac{1}{n}$. This is the harmonic series which we know to be divergent.

$$\lim_{n\to\infty} \frac{\left(n^2 - 1\right)/n^3}{1/n} = \lim_{n\to\infty} \frac{n\left(n^2 - 1\right)}{n^3} = \lim_{n\to\infty} \frac{n^3 - n}{n^3} = 1$$

This limit is positive and finite, so we conclude by the limit comparison test that $\sum_{n=2}^{\infty} \frac{n^2 - 1}{n^3}$ diverges. (We could also have used direct comparison in this case. Try it!)

Section 7 Problems

In Problems 1-10, determine whether the given series converges by using either the integral test or the *p*-series test.

1. $\frac{1}{1}+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\frac{1}{625}+\cdots$
2. $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\frac{1}{1024}+\cdots$
3. $\frac{1}{e}+\frac{2}{e^4}+\frac{3}{e^9}+\frac{4}{e^{16}}+\frac{5}{e^{25}}+\cdots$
4. $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$
5. $\sum_{n=1}^{\infty}\frac{3}{\sqrt[5]{n}}$
6. $\sum_{n=1}^{\infty}\frac{1}{n^e}$
7. $\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$
8. $\sum_{n=1}^{\infty}\frac{n^2}{\left(n^3+2\right)^4}$
9. $\sum_{n=1}^{\infty}\frac{3n^{1/3}}{2n^{2/5}}$
10. $\sum_{n=0}^{\infty}\frac{1}{2n+5}$

In Problems 11-19, determine whether the given series converges by using one of the comparison tests.

11. $\sum_{n=0}^{\infty}\frac{n^3+2n}{n^5+8}$
12. $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}-1}$
13. $\sum_{n=3}^{\infty}\frac{1}{\ln(\ln n)}$
14. $\sum_{n=0}^{\infty}\frac{3^n}{4^n+2}$
15. $\sum_{n=0}^{\infty}\frac{2^n-5}{n!}$
16. $\sum_{n=0}^{\infty}\frac{3^n+2}{4^n}$
17. $\sum_{n=2}^{\infty}\frac{n^3-1}{n^4+1}$
18. $\sum_{n=0}^{\infty}\frac{1}{2n+4^n}$
19. $\sum_{n=0}^{\infty}\frac{1}{an+b}$, where a is positive.

In Problems 20-33, determine whether the series converges by using any of the tests from this chapter.

20. $\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\frac{8}{81}+\frac{16}{243}+\cdots$
21. $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\cdots$
22. $\sum_{n=1}^{\infty}\frac{3^n\cdot n^3}{n!}$
23. $\sum_{n=3}^{\infty}4\left(\frac{2}{7}\right)^{n-2}$
24. $\sum_{n=0}^{\infty}\frac{n^2+2^n}{n!}$
25. $\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$
26. $\sum_{n=1}^{\infty}\cos\left(\frac{1}{n}\right)$
27. $\sum_{n=1}^{\infty}n\sin\left(\frac{1}{n}\right)$
28. $\sum_{n=1}^{\infty}n\cos\left(\frac{1}{n}\right)$
29. $\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{1}{n}\right)$
30. $\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(\frac{1}{n}\right)$
31. $\sum_{n=0}^{\infty}\frac{3^n\cdot n!}{(n+1)!}$

32. $\sum_{n=2}^{\infty}\left(\frac{1}{n^3}-\frac{1}{n^4}\right)$

33. $\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^2}$

In Problems 34-40, use each of the following tests *once* to determine whether the given series converge: n^{th} term test, geometric series test, ratio test, integral test, p-series test, direct comparison test, limit comparison test.

34. $\sum_{n=0}^{\infty}\frac{1}{2n+1}$

35. $\sum_{n=1}^{\infty}n$

36. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$

37. $\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n$

38. $\sum_{n=0}^{\infty}\frac{n!}{3^n}$

39. $\sum_{n=2}^{\infty}\frac{3}{n^2-1}$

40. $\sum_{n=1}^{\infty}\frac{e^n}{n}$

In Problems 41-49, use each of the following tests *once* to determine whether the given series converge: n^{th} term test, geometric series test, telescoping series test, ratio test, root test, integral test, p-series test, direct comparison test, limit comparison test.

41. $\sum_{n=3}^{\infty}\frac{n}{n^2}$

42. $\sum_{n=3}^{\infty}\frac{n}{n^2-4}$

43. $\sum_{n=3}^{\infty}\frac{n}{n^2+4}$

44. $\sum_{n=0}^{\infty}\left(\frac{2}{n+2}-\frac{2}{n+1}\right)$

45. $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}\ln n}$

46. $\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n$

47. $\sum_{n=1}^{\infty}\frac{n^2}{3^n}$

48. $\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^n$

49. $\sum_{n=0}^{\infty}\left(\frac{3}{2n+1}\right)^n$

50. Use the ratio test to attempt to determine the convergence of the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt[4]{n}}$ and $\sum_{n=1}^{\infty}\frac{1}{n^3}$. Do these series converge or diverge? What important fact about the ratio test is being confirmed here?

51. Suppose p is a positive number. Under what further conditions on p does the series $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$ converge?

52. Suppose p is a positive number. Under what further conditions on p does the series $\sum_{n=2}^{\infty}\frac{1}{n^p\ln n}$ converge?

53. Suppose p is a positive number. Under what further conditions on p does the series $\sum_{n=2}^{\infty}\frac{\ln n}{n^p}$ converge?

54. Let p_n be the n^{th} prime number (i.e., $p_1=2$, $p_2=3$, $p_3=5$, etc.). The Prime Number Theorem is a theorem that discusses the distribution of the numbers p_n. One consequence of the Prime Number Theorem is that $\lim_{n\to\infty}\frac{p_n}{n\ln n}=1$. (We say that the primes are "asymptotic to" $n\ln n$ and we write $p_n\sim n\ln n$ for this relationship. It means that the n^{th} prime is roughly equal to $n\ln n$, at least on a relative scale.) Use this

fact to determine whether the series $\sum_{n=1}^{\infty}\frac{1}{p_n}$ converges.

55. Let f_n be the n^{th} Fibonacci number. (The Fibonacci numbers are the sequence 1, 1, 2, 3, 5, 8, 13, 21, … where $f_n = f_{n-1} + f_{n-2}$.) It is known that the ratio of successive Fibonacci numbers approaches the Golden Ratio ϕ, which is roughly 1.618. In symbols, $\lim_{n\to\infty}\frac{f_n}{f_{n-1}} = \phi$. Determine whether the series $\sum_{n=1}^{\infty}\frac{1}{f_n}$ converges.*

56. Suppose $p_1(x)$ and $p_2(x)$ are polynomial functions that are both positive for $x \geq N$. Under what conditions on the degrees of $p_1(x)$ and $p_2(x)$ will $\sum_{n=N}^{\infty}\frac{p_1(n)}{p_2(n)}$ converge?

57. Give an example of two divergent series $\sum a_n$ and $\sum b_n$ such that $\sum \frac{a_n}{b_n}$ converges.

58. Give an example of two convergent series $\sum a_n$ and $\sum b_n$ such that $\sum \frac{a_n}{b_n}$ diverges.

59. Give an example of two series $\sum a_n$ and $\sum b_n$ such that $\sum b_n$ diverges, $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$, and $\sum a_n$ converges.

60. Give an example of two series $\sum a_n$ and $\sum b_n$ such that $\sum b_n$ diverges, $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$, and $\sum a_n$ diverges.

61. In the spirit of Problems 60 and 61, can you come up with two series $\sum a_n$ and $\sum b_n$ such that $\sum b_n$ converges, $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$, and $\sum a_n$ diverges?

62. Reconsider Problems 59 through 61, but this time suppose that $\lim_{n\to\infty}\frac{a_n}{b_n} = \infty$. Under what conditions can you infer the convergence behavior of one series from the behavior of the other?

63. Suppose that for some sequence of positive numbers $\{a_n\}$, $\lim_{n\to\infty} na_n = L$, where L is finite, positive, and non-zero. Show that $\sum a_n$ diverges. (Hint: Compare to the harmonic series.)

64. Suppose that $\sum \frac{a_n}{n}$ converges ($a_n > 0$). Must it be true that $\sum a_n$ converges?

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* In this chapter, we have looked at series convergence being a question of how quickly the terms of a series approach zero. Another perspective is to consider "sparseness." We know that the sum 1 + 1/2 + 1/3 + … diverges, but if we take out enough of the terms, say all the ones whose denominators are not perfect squares, we obtain $\Sigma 1/n^2$, a series that converges. One way to think of this is to say that the perfect squares are sufficiently few and far between—sufficiently sparse—that the series of their reciprocals converges. This problem and the previous one can be interpreted as telling us something about the sparseness of the primes and the Fibonacci numbers.

Section 8 – Varying-Sign Series

So far we have almost exclusively considered series in which all terms were positive (or at least non-negative). The only exceptions to this have been geometric series with $r < 0$ and the times that we have cheated by ignoring sign variation (as in using the ratio test to determine the radius of convergence of a power series). It is time to break free of this constraint and examine series that include negative terms. Among other things, this will put our initial explorations of power series on more solid ground.

The simplest next step is to consider series in which all the terms are negative (or at least non-positive). Of course, this isn't really a change. If we just factor the negative out of the summation, we are left with a positive-term series: $\Sigma(-a_n) = -\Sigma a_n$. If this is the case, we can apply all the tests we have seen in recent sections.

Eventually, though, we have to consider series in which the signs of the terms vary. The worst possible situation is if the signs vary irregularly. In many cases, such as with $\sum_{n=1}^{\infty}\frac{\sin n}{n}$, determining convergence will be outside the scope of this chapter,* though we will sometimes be able to puzzle out whether such series converge.

We will begin with the simplest case of varying signs: the case where the terms strictly alternate. An example (an important one, it turns out) is $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$. Such series are called "alternating series."

Alternating Series and the Alternating Series Test

Definition:	An **alternating series** is a series in which the terms strictly alternate in sign. In other words, no two consecutive terms have the same sign.

Alternating series are often represented in the form $\sum(-1)^n a_n$ or $\sum(-1)^{n+1}a_n$. When written this way, we can parse the terms of the series into an alternating factor ($(-1)^n$, or something similar), and a factor that is positive (a_n) and represents the magnitude of the term. At other times, though, we will let a_n stand for the entire term with the alternating factor wrapped into it. Hopefully, context will make it clear what a_n means.

Practice 1
Which of the following series are alternating?

a. $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n!}$ b. $\sum_{n=0}^{\infty}(-3)^n$ c. $\sum_{n=0}^{\infty}\frac{\cos(n\pi)}{n^2+1}$ d. $\sum_{n=1}^{\infty}\sin(n)$

An alternating series can be either convergent or divergent. In Practice 1, the series in (a) and (c) converge, while series (b) diverges.

In order to gain an understanding of convergence for alternating series, let's look at the one mentioned above:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots+(-1)^{n+1}\cdot\frac{1}{n}+\cdots.$$

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* You can research the Dirichlet and Abel tests if you are interested.

This series is called the alternating harmonic series. It is just like the harmonic series that we have seen over and over again since Section 1, but the terms alternate. Recall that the regular harmonic series diverges; although the terms go to zero, they do not do so quickly enough for convergence. What about the alternating harmonic series? The table below shows several partial sums s_n of the series. (In this table, a_n represents the term of the series, sign included. It is not only the magnitude.)

n	1	2	3	4	5	6	7	8	9	10
a_n	1	-0.5	0.3333	-0.25	0.2	-0.8333	0.1429	-0.125	0.1111	-0.1
s_n	1	0.5	0.8333	0.58333	0.7833	0.6167	0.7595	0.6345	0.7456	0.6456

Do you see what's happening? The partial sums are bouncing up and down around some number, but all the while they are zeroing in on something. Maybe it will be clearer if we look a bit farther out.

n	100	101	102	103	104	105	106	107	108
a_n	-0.01	0.0099	-0.0098	0.0097	-0.0096	0.0095	-0.0094	0.0094	-0.0093
s_n	0.6882	0.6981	0.6883	0.6980	0.6884	0.6979	0.6885	0.6978	0.6885

Now can you see how the partial sums are drifting in towards some number? In fact, the alternating series converges to about 0.6931. The tables give some indication of how this happens. As we add up the terms, adding a positive term will overshoot the target of 0.6931, but then it will be followed by a negative term which will bring it back down. But the negative term overshoots as well. There is a constant give-and-take between the successive positive and negative terms. But there is more to it because the terms are also decreasing in size throughout. So while adding each term pushes the partial sum in the opposite direction as the previous one did, it does so by less and less each time. Put another way, each successive term partially cancels the effect of the one before it, but never fully. As a consequence, we see the partial sums gradually drift towards a limit.

Figure 8.1 is an attempt to make this clearer. The blue dots represent the terms a_n of the alternating harmonic series (sign and all) as a function of n. The red dots represent the corresponding partial sums s_n. (Note that $a_1 = s_1$, so only one dot is visible on the graph for n = 1.) As you can see, the blue dots are "funneling in" toward zero, confirming that the series passes the n^{th} term test. The partial sums, for their part, are tending toward the limit $s = 0.6931$. Look at the graph one n-value at a time. A blue dot below the n-axis corresponds to a red dot below the dotted line; the negative term has pulled the partial sum below its eventual limit. This is followed, though, by a blue dot above the n-axis; at the same time the partial sum dot has drifted back above 0.6931, but not as far as it had been. The picture tries to show the partial cancellation—the give and take between positive and negative terms as they pull on the partial sums—in action. This is why the series converges, even though the terms are not approaching zero particularly quickly.

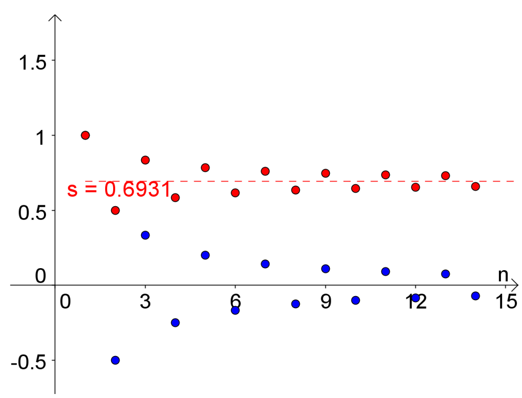

Figure 8.1: Terms and partial sums of the alternating harmonic series

The above discussion identified two key factors responsible for the alternating harmonic series converging. First and most fundamentally, the terms alternate in sign; this allows them to cancel partially. Second, the terms are decreasing in size so that the cancellation is never complete; this is why the oscillation in the partial sums is damped, leading to a well-defined limit. And of course it goes without saying that the terms of the series approach zero. If they did not, the series would not pass the n^{th} term test and the series would diverge.

We summarize these observations in our final formal convergence test.

Theorem 8.1 – The Alternating Series Test
If a_n is positive, the series $\sum(-1)^n a_n$ (or $\sum(-1)^{n+1} a_n$, etc.) converges if $\lim\limits_{n\to\infty} a_n = 0$ and if $a_{n+1} < a_n$ for all n (at least past a threshold N).
Put another way, a series converges if the terms

1. strictly alternate,
2. decrease in magnitude, and
3. tend to zero.

Example 1
Use the alternating series test (AST) when applicable to determine which of the series in Practice 1 converge.
Solution

a. The series is $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n!}$. We will check the three criteria from Theorem 8.1. First, the terms of the series can be written as $(-1)^{n+1}\cdot a_n$ where $a_n = \frac{1}{n!}$, so this series is alternating. Second, for all n greater than 0, $\frac{1}{(n+1)!} < \frac{1}{n!}$. This shows that $a_{n+1} < a_n$. (Notice that here we needed to use our threshold N. If $n = 0$, it is *not* true that $\frac{1}{(n+1)!} < \frac{1}{n!}$. Fortunately, the decreasing behavior sorts itself out quickly.) Finally, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{1}{n!} = 0$, as required. This series converges by the AST.

b. The series is $\displaystyle\sum_{n=0}^{\infty}(-3)^n$. This series is alternating; the terms can be written as $(-1)^n\cdot 3^n$. However, the sequence $a_n = 3^n$ is not decreasing. Therefore the hypotheses of the AST do not apply to this series. We can conclude nothing from the AST. (Other tests, namely the geometric series test and n^{th} term test, can be used to show that this series diverges.)

c. The terms of the series in part (c) can be written as $(-1)^n\cdot a_n$ where $a_n = \frac{1}{n^2+1}$, so the series is alternating. $a_{n+1} < a_n$ since $\frac{1}{(n+1)^2+1} < \frac{1}{n^2+1}$ for all n. Finally, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{1}{n^2+1} = 0$. By the AST, $\displaystyle\sum_{n=0}^{\infty}\frac{\cos(n\pi)}{n^2+1}$ converges.

d. We must pass on the series in part (d). The terms of this series do not strictly alternate, so the AST has nothing to tell us about this series. (The n^{th} term test tells us that it diverges.) ◊

Notice that there is no divergence condition for the alternating series test. I will say that again. THE AST CANNOT BE USED TO SHOW DIVERGENCE. This is a sticking point for many students just as using the n^{th} term test to show convergence can be. Speaking of the n^{th} term test, if a series fails criterion 3 in Theorem 8.1, then the series does diverge, but it is not because of the AST. Number 3 in the theorem is simply a restatement of the n^{th} term test. Most of the time when you are tempted to say, "The series diverges because of the AST," you *really* want to say, "The series diverges because it does not pass the n^{th} term test." Most of the time.

Practice 2
Use the alternating series test, if applicable, to determine whether the following series converge.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ b. $\displaystyle\sum_{n=2}^{\infty}(-1)^n\cdot\frac{n}{n-1}$ c. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n(n+1)/2}\cdot\frac{1}{n}$

I know what you're thinking. You are thinking criteria 2 and 3 from Theorem 8.1 are redundant. Cleary if the terms tend to zero, their absolute values must be decreasing. Well, no. It is true in a certain sense that if the terms get closer to zero, then in the long run the terms will get smaller. But that tells us nothing about the relative size of *consecutive* terms. Remember that it is the relationship of consecutive terms that ensures the convergence we see in the AST. Some standard examples to help us see this are in Example 2.

Example 2
Provide a divergent and a convergent alternating series in which $\lim_{n\to\infty}a_n=0$, but in which the terms do not strictly decrease in magnitude.
Solution
Consider the series

$$\frac{1}{1}-\frac{1}{4}+\frac{1}{3}-\frac{1}{16}+\frac{1}{5}-\frac{1}{36}+\frac{1}{7}-\frac{1}{64}+\cdots.$$

In summation notation, this is $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ where $a_n=\frac{1}{n}$ if n is odd and $a_n=\frac{1}{n^2}$ if n is even. The terms in this series alternate in sign and do approach zero, but there is no N for which $a_{n+1}<a_n$ for all n beyond N. In fact, this series diverges.

As another example, consider

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{2^2}-\frac{1}{4^2}+\frac{1}{2^3}-\frac{1}{4^3}+\cdots.$$

This series is formed by inter-weaving two convergent geometric series, one with positive terms and the other with negative. This series also does not show monotonic decrease towards zero in the magnitude of the terms. In this case, though, the series converges. ◊

Moral: All three parts of the AST are essential to applying Theorem 8.1.

Alternating Series Error Bound

The nature of convergence that we saw in the alternating series test tells us about how closely the n^{th} partial sum approximates the value of the series. Since adding each term overshoots the actual sum, but by ever-diminishing amounts, the error after n terms can never be larger than the magnitude of the next term a_{n+1}. This is because adding a_{n+1} to s_n will move the partial sum in the direction towards the actual sum of the series, but will go too far; part of its magnitude corrects for the error in s_n, while the rest of its magnitude is left over as new error in s_{n+1}.

The table that follows, again for the alternating harmonic series, shows the same data you have already seen, but it also includes the actual error in the n^{th} partial sum (shown as a magnitude, so always positive), as well as the absolute value of the "next" term. As with the previous table, a_n is used to represent the entirety of the term, not just the positive factor. As you can clearly see, the magnitude of a_{n+1} is always greater than the error in s_n. In this case, this error estimate seems to be fairly conservative; the actual error is quite a bit less than the size of the next term for this series.

n	1	2	3	4	5	6	7	8	9	10
a_n	1	-0.5	0.3333	-0.25	0.2	0.8333	0.1429	-0.125	0.1111	-0.1
s_n	1	0.5	0.8333	0.58333	0.7833	0.6167	0.7595	0.6345	0.7456	0.6456
Actual Error	0.3069	0.1931	0.1402	0.1098	0.0902	0.0765	0.0664	0.0586	0.0525	0.0475
$\lvert a_{n+1}\rvert$	0.5	0.3333	0.25	0.2	0.8333	0.1429	0.125	0.1111	0.1	0.0909

We summarize these observations in a theorem.

Theorem 8.2 – Alternating Series Error Bound
If $\sum a_n$ is a convergent alternating series in which $\lvert a_{k+1}\rvert < \lvert a_k\rvert$ for all k (at least past some threshold), then the error in the n^{th} partial sum is no larger than $\lvert a_{n+1}\rvert$.

Another way of looking at this is that for alternating series whose terms decrease monotonically in magnitude, the actual value of the series will always fall between any two consecutive partial sums. Go back to the table above and pick any two partial sums that are next to each other. Notice that the actual value of the series, about 0.6931, is indeed between those two partial sums. Now take a moment to convince yourself that this really does follow from Theorem 8.2; if the error in s_n is no more than $\lvert a_{n+1}\rvert$, then the actual value of the series must fall somewhere between s_n and s_{n+1}.

Notice that Theorem 8.2 only comes into play if the terms of the series are monotonically decreasing in size. Thus, Theorem 8.2 does not apply to series like those in Example 2. Fortunately, most alternating series are not like the ones in Example 2.

Example 3

Use s_5 and Theorem 8.2 to give bounds on the value of $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$.

Solution

Your calculator will tell you that $s_5 = 0.3\overline{6}$. Furthermore, $\lvert a_6\rvert = 0.0013\overline{8}$. Therefore, the actual value of the series is somewhere between $0.3\overline{6} - 0.0013\overline{8}$ and $0.3\overline{6} + 0.0013\overline{8}$. To four decimal places, we conclude that $0.3653 \le \sum_{n=0}^{\infty}\frac{(-1)^n}{n!} \le 0.3681$. The actual value of this series, as we will be able to see in Section 10, is $1/e$, or 0.3679. This is indeed within our bounds.

(It would also be correct to say that since $s_5 = 0.3\overline{6}$ and $s_6 \approx 0.3861$, then $0.3\overline{6} \le \sum_{n=0}^{\infty}\frac{(-1)^n}{n!} \le 0.3861$; the value of the series must fall between two consecutive partial sums. These are actually better bounds than those we found originally since they trap the value of the series in a tighter interval. Even so, I tend to stick to the approach above because of its similarity to the way we handled Lagrange error bounds in Section 4.) ◊

Practice 3

Earlier, we claimed that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is roughly 0.693. How many terms are needed in the series to guarantee that the approximation will be accurate to three decimal places?

It is time to take a quick trip back a few sections to revisit the world of Taylor polynomials. This is an awkward mix of looking back while simultaneously getting ahead of ourselves, but that's okay. In

Section 4 we discussed how to use Lagrange remainder to estimate the error involved in a polynomial approximation. For example, to estimate the error involved in using the third-degree Maclaurin polynomial for the sine function to compute sin(1), we examined the quantity $\frac{M}{4!}(1-0)^4$. For the sine function, we could use $M = 1$ because 1 is sure to be an upper bound for any derivative of the sine function on any interval. Thus, we concluded that $\sin(1) \approx 1 - \frac{1^3}{3!} = 0.8\overline{3}$ with an error of no more than $\frac{1}{4!}$ or 1/24.

But we can also use the alternating series error bound, as Example 4 shows.

Example 4
Use the alternating series bound to estimate the amount of error involved in approximating sin(1) with a 3rd-degree Macluarin polynomial.
Solution
We know that the $(2n+1)^{\text{st}}$-order Maclaurin polynomial evaluated at $x = 1$ gives

$$1 - \frac{1^3}{3!} + \frac{1^5}{5!} - \cdots + (-1)^n \frac{1^{2n+1}}{(2n+1)!}.$$

This is "just" an alternating series, or at least the truncation of one. (This is how we're getting ahead of ourselves. We have not yet defined a Taylor series even though we looked at this particular one in depth at the end of Section 6.) The error, then, is bounded by the first omitted term in the series. For the approximation $\sin(1) \approx 1 - \frac{1^3}{3!}$, the error is no more than the next term that would be in the series: $\frac{1}{5!}$. Therefore we conclude that the maximum error in our approximation of sin(1) is 1/120. ◊

The point of Example 4 is not that the alternating series error bound is tighter than the Lagrange error bound we obtained in the previous paragraph. That will not always be the case. The point is that when we are lucky enough to be dealing with an alternating series, the alternating series error bound is much easier to obtain than the Lagrange error bound.

Practice 4
Using the alternating series error bound, what degree Maclaurin polynomial is guaranteed to estimate cos(3) with an error of no more than 0.001?

Absolute and Conditional Convergence

We move on to the question of convergence for series whose terms vary in sign without strictly alternating. We will only give a partial answer to this question, and before we do we need a little more vocabulary.

Definition: A series $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ is convergent. A series that is convergent, but not absolutely convergent, is **conditionally convergent**. In other words, if $\sum a_n$ converges but $\sum |a_n|$ diverges, then $\sum a_n$ is conditionally convergent.

To see why we make a distinction between absolutely and conditionally convergent series, consider two examples: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. As we have seen, the first series, the alternating harmonic series, converged because of a partial cancelling of consecutive terms. There was a delicate balance between the positive and the negative terms that made the partial sums hover around a fixed value. But if we take the alternation away, we are left with the regular harmonic series, a series for which the terms do not go to zero fast enough for convergence. That is, if we consider $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$, we replace the convergent series that we are interested in with a series that diverges. Therefore, we say that the alternating harmonic series converges conditionally.

However, the situation is very different with $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$. This series surely converges by the AST. You are welcome to work out the details for yourself. From one perspective, this means that the positive and negative terms cancel out well enough for convergence. But there is something else going on with this series. Even if the terms did not alternate, they would still be approaching zero very quickly. The size of the terms is given by $1/n^2$, and we already know that these numbers decay quickly. Put another way, if we look at $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$, we have a convergent p-series. This is absolute convergence.

The use of the word "absolute" is not an accident. An absolutely convergent series is one where the series formed from the *absolute* values of the terms still converges. But this is just about the symbols involved. We can also consider this idea on a more conceptual level. Conditionally convergent series converge only by the grace of cancelling between the positive and negative terms; the terms do not really go to zero fast enough, but by good fortune the series still converges. In the case of absolute convergence it is almost as though the series does not care what sign the terms have. The terms are vanishing fast enough that it just doesn't matter.

Absolute convergence is a stronger form of convergence than conditional convergence because we can tinker with the signs of the terms in an absolutely convergent series without affecting the convergence. Absolutely convergent series are robust in this way, while conditionally convergent series are more delicate. Since absolute convergence is such a strong form of convergence, any absolutely convergent series is convergent in the regular sense (i.e., in the sense that the limit of the partial sums exists). This is a theorem, and it can be proved. But the idea is more important for us than the proof. Absolutely convergent series are *really* convergent. When we state the theorem, it might seem self-evident to you, as if it is just a game with words. That's okay for now. We still state the theorem for reference.

Theorem 8.3
If a series converges absolutely, then the series converges.

If possible, we prefer to work with absolutely convergent series, though we do not always have the luxury to restrict our horizons this way. Absolutely convergent series behave better than conditionally convergent series, and they generally work exactly the way we expect. In Section 9 we will explore some of the weird and wild things that can happen with conditionally convergent series. Notice that a positive-term series that is convergent must be absolutely convergent; replacing the terms with their absolute values would not actually change anything.

Example 5
Determine whether the following series are absolutely convergent, conditionally convergent, or divergent.

a. $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$ b. $\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n}+1}$ c. $\sum_{n=0}^{\infty}(-1)^n\cdot\frac{n+3}{n+4}$

Solution

a. This series converges by the AST. In essence, this was shown in Example 1. The signs have all flipped relative to the earlier example, but that will not affect the use of the AST. To see whether the convergence is absolute or conditional, we consider $\sum_{n=0}^{\infty}\left|\frac{(-1)^n}{n!}\right|$ which is $\sum_{n=0}^{\infty}\frac{1}{n!}$. We have seen (multiple times) that this series converges. Therefore, $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$ converges absolutely.

b. This series is alternating, as can be seen by writing the terms as $(-1)^n a_n$ where $a_n = \frac{1}{\sqrt{n}+1}$. $\frac{1}{\sqrt{n+1}+1} < \frac{1}{\sqrt{n}+1}$ for all $n \geq 0$. Finally, $\lim_{n\to\infty}\frac{1}{\sqrt{n}+1} = 0$. Therefore the series converges by the AST. Now we look at the absolute value series $\sum_{n=0}^{\infty}\left|\frac{(-1)^n}{\sqrt{n}+1}\right| = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n}+1}$. This series diverges, as can be shown by using limit comparison against $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ (a divergent *p*-series: $p = \frac{1}{2} \leq 1$). Briefly, $\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n}+1}}{\frac{1}{\sqrt{n}}} = 1$, which is positive and finite. Since the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n}+1}$ converges, but the corresponding absolute value series diverges, the original alternating series is conditionally convergent.

c. The n^{th} term test shows us that this series diverges: $\lim_{n\to\infty}(-1)^n \cdot \frac{n+3}{n+4} \neq 0$. Period. End of story. ◊

One cool thing about absolute convergence is that we can use it to determine convergence of some series without actually exploring the series directly.

Example 6

Show that the series $\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ converges.

Solution

It is tempting to use some kind of comparison test against $\sum\frac{1}{n^2}$, but this would not be justified. The given series is not a positive-term series so comparison tests are off the table. The series does not strictly alternate either, so the AST does not apply. None of our convergence tests are of any use here. However, let us consider the corresponding absolute value series $\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{|\cos n|}{n^2}$. This series *is* a positive-term series so we can use all our tests to decide its convergence. Since $|\cos n| \leq 1$ it follows that $\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$ for all *n*. The series $\sum\frac{1}{n^2}$ is a convergent *p*-series ($p = 2 > 1$), so the series $\sum_{n=1}^{\infty}\frac{|\cos n|}{n^2}$ converges by direct comparison to $\sum_{n=1}^{\infty}\frac{1}{n^2}$. This means, in turn, that $\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ converges absolutely. By Theorem 8.3, the given series converges. ◊

Power Series Concluded

We have learned all the convergence tests that we are going to learn in this course, and it is time to finally return to the topic of central importance to us in this chapter: power series and their intervals of convergence. The ratio test is typically used to get us started, but the tests from this section and Section 7 are used for determining convergence at the endpoints of the intervals. Let's dive in.

Example 7

Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n \cdot 3^n}$.

Solution

We begin, as we did in Section 6, by applying the ratio test to the general term of the series. But recall that we always actually applied the ratio test to the *absolute value* of the general term. When we first started doing this, it probably seemed like cheating. But now that we have worked through this section, we can see that we are actually finding an interval on which the series converges *absolutely*.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{(n+1)\cdot 3^{n+1}}\cdot\frac{n\cdot 3^n}{(x-5)^n}\right| &= \lim_{n\to\infty}\left|\frac{n}{(n+1)}\cdot\frac{3^n}{3^{n+1}}\cdot\frac{(x-5)^{n+1}}{(x-5)^n}\right| \\
&= \lim_{n\to\infty}\frac{n}{(n+1)}\cdot\frac{1}{3}\cdot|x-5| \\
&= \frac{|x-5|}{3}
\end{aligned}$$

In order for the ratio test to give convergence, we need this limit to be less than 1.

$$\begin{gathered}
\frac{|x-5|}{3} < 1 \\
|x-5| < 3 \\
-3 < x-5 < 3 \\
2 < x < 8
\end{gathered}$$

Therefore our "rough draft" interval of convergence is $2 < x < 8$. What we have actually shown is that the power series converges absolutely for these x-values. Therefore, by Theorem 8.3, the power series converges on this interval. (By way of reminder, I'll point out that this interval is symmetric about the center of the series, $x = 5$.)

Now we have to consider the endpoints: $x = 2$ and $x = 8$. These must be checked individually since they need not exhibit the same behavior as one another.

At $x = 8$, the power series is $\sum_{n=1}^{\infty}\frac{(8-5)^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{3^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{1}{n}$. This is the harmonic series, which we know diverges. The power series diverges when $x = 8$.

At $x = 2$, the power series is $\sum_{n=1}^{\infty}\frac{(2-5)^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{(-3)^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n\cdot 3^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$. This is the alternating harmonic series, and we know that it converges conditionally. (Actually, the signs are all wrong; the alternating series starts with a positive term. But this does not affect convergence.) Therefore the power series converges at $x = 2$.

We put all the information together to obtain our final interval of convergence: $2 \le x < 8$. The power series converges for all x-values in this interval, while it diverges for all x-values outside of this interval. This is the domain of the power series. It is the set of x-values for which the series makes sense. ◊

Practice 5

Find the interval of convergence of the power series $\sum_{n=0}^{\infty}\frac{(3x)^n}{4^n+1}$.

To summarize our understanding of power series so far…

1. Every power series has an interval of convergence. These are the *x*-values for which the series converges and for which we can get sensible values out of the infinite series.
2. Every power series has a radius of convergence. The radius of convergence can be 0 (the power series converges only at its center), infinite (the power series converges for all *x*), or a positive finite number. In the latter case, the interval of convergence is an interval symmetric about its center (give or take the endpoints). The width of the interval is twice the radius of convergence.
3. The power series always converges absolutely in the interior of the interval of convergence. This is usually determined by the ratio test, but if the power series is geometric, then we can use the geometric series test. To determine whether the series converges (conditionally or absolutely) or diverges at its endpoints, we use the convergence tests that we have seen in this chapter.
4. We can integrate or differentiate a power series term by term to produce a new power series. The new power series will have the same radius of convergence (and center, of course), but its behavior at the endpoints may change. When we differentiate, we might lose endpoints from the interval of convergence. Conversely, when we integrate we may gain endpoints. There is no general way to predict what will happen; you just have to apply convergence tests to the endpoints of the new series.

In Section 10 we will use these facts in relation to power series that are specifically *Taylor* series. Doing so will complete our study of series. But let's take a little preview…

At the end of Section 6 we claimed that the power series $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{x^n}{n}$ equals $g(x)=\ln(1+x)$, at least on the interval of convergence of the series. The center of this power series is $x=0$, and we saw in Section 6 that its radius of convergence is 1. We are finally in a position to determine whether the power series converges at its endpoints.

x = -1: The series evaluates to $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{(-1)^n}{n}=\sum_{n=1}^{\infty}\frac{-1}{n}=-\sum_{n=1}^{\infty}\frac{1}{n}$. This is the (opposite of the) harmonic series. It diverges.

x = 1: The series is now $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$. This is the alternating harmonic series. By the AST, it converges.

We can now say once and for all that the interval of convergence of this power series, which we will call the *Taylor series* for $g(x)$, is $-1<x\le 1$. The Taylor series represents the function $g(x)=\ln(1+x)$ on this interval. A couple quick checks are in order. First, notice that it is a very reassuring thing that the power series diverges when $x=-1$. $g(-1)=\ln 0$ which is undefined. The graph of g has a vertical asymptote at this point, and the interval of convergence of the power series stops just as its *x*-values teeter on the brink.

At the other end of the interval, x = 1 is included in the interval of convergence. This means that $g(1)=\ln(1+1)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$. This tells us more than the simple fact that the alternating harmonic series converges; *it tells us to what value the series converges*. The alternating harmonic series converges to ln(2). Recall, though, that earlier in this section I claimed that the alternating harmonic series converged to a value around 0.6931, and I presented numerical data to support that claim. Pick up your calculator and punch in ln(2) to see what the decimal approximation is. Go ahead; I'll be right here.

Pretty cool, right?

Answers to Practice Problems

1. All are alternating series except for (d). This should be clear for (a) and (b) if you simply write out a few terms. In (c) the alternating factor has been disguised slightly, but if you write out a few terms of $\cos(n\pi)$ and evaluate the trig expressions, you will see that this factor is indeed responsible for making the terms alternate in sign. The final series displays irregular sign variation. For a while it looks like there is a simple pattern to the signs: three positive, three negative, etc. But if you follow this series long enough, this pattern will break down. Even if the pattern held, though, we would not call the series alternating because the signs do not change with every term.

2. For part (a), which is an alternating series, take $a_n = \frac{1}{\sqrt{n}}$. $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ for all n. This series converges by the AST.
In part (b), the series is alternating, but $\lim_{n\to\infty} \frac{n}{n-1} = 1 \neq 0$. The alternating series test does not apply. The n^{th} term test, however, tells us that this series diverges.
The series in part (c) is not alternating. Write out a few terms to see this. The signs of the terms follow a clear pattern, but it is not one of strict alternation. We can conclude nothing about this series based on the AST. (It turns out that this series converges.)

3. To be accurate to three decimal places is to have error less than 0.0005. Therefore, we want $|a_{n+1}| < 0.0005$ or, simply, $\frac{1}{n+1} < 0.0005$. Solving for n gives $n > 1999$. Thus s_{2000} is guaranteed to have the required accuracy. (This series converges much more slowly than the one in Example 3.)

4. From our study of Maclaurin polynomials we know that

$$\cos(3) \approx 1 - \frac{3^2}{2!} + \frac{3^4}{4!} - \frac{3^6}{6!} + \cdots + (-1)^n \cdot \frac{3^{2n}}{(2n)!}.$$

The error in this estimation is no more than the next term in the series—the first term we did not use. In this case that is, in absolute value, $\frac{3^{2n+2}}{(2n+2)!}$. Using a calculator to scan a table of values, we find that $\frac{3^{2n+2}}{(2n+2)!} < 0.001$ when $n \geq 6$. This means that as long as we use an n value of at least 6, we will have the desired accuracy. But be careful. The degree of the polynomial being used is $2n$, not n. Therefore, we need a 12^{th}-degree Maclaurin polynomial to approximate cos(3) with error less than 0.001. The calculator value of cos(3) is about -0.98999, while $P_{12}(3) = -0.98994$, so we see that the polynomial approximation is well within the required tolerance.

5. We begin, as always, with the ratio test applied to the absolute value of the general term.

$$\lim_{n\to\infty} \left| \frac{(3x)^{n+1}}{4^{n+1}+1} \cdot \frac{4^n+1}{(3x)^n} \right| = \lim_{n\to\infty} \frac{4^n+1}{4^{n+1}+1} \cdot \frac{3^{n+1}}{3^n} |x| = \frac{3}{4} \cdot |x|$$

We require that $\frac{3}{4}|x| < 1$ or $|x| < \frac{4}{3}$. The rough draft interval of convergence is $-\frac{4}{3} < x < \frac{4}{3}$. Now we must check endpoints.

$x = \frac{4}{3}$: The series is $\sum_{n=0}^{\infty} \frac{\left(3 \cdot \frac{4}{3}\right)^n}{4^n+1} = \sum_{n=0}^{\infty} \frac{4^n}{4^n+1}$. This series diverges by the n^{th} term test: $\lim_{n\to\infty} \frac{4^n}{4^n+1} = 1 \neq 0$. Therefore the power series diverges at $x = \frac{4}{3}$.

$x = -\frac{4}{3}$: Now the series is $\sum_{n=0}^{\infty} \frac{\left(3 \cdot \frac{-4}{3}\right)^n}{4^n + 1} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^n}{4^n + 1}$. The alternation does not help the terms go to zero. This series still diverges by the n^{th} term test. The power series diverges at $x = -\frac{4}{3}$.

The interval of convergence for the power series is $-\frac{4}{3} < x < \frac{4}{3}$.

Section 8 Problems

In Problems 1-15, determine whether the given series converges absolutely, converges conditionally, or diverges.

1. $\frac{1}{1}+\frac{-1}{8}+\frac{1}{27}+\frac{-1}{64}+\frac{1}{125}+\cdots$
2. $\frac{2}{1}-\frac{3}{4}+\frac{4}{9}-\frac{5}{16}+\frac{6}{25}-\cdots$
3. $\sum_{n=0}^{\infty} 3\cdot\left(\frac{-1}{2}\right)^n$
4. $\sum_{n=1}^{\infty} (-1)^n\cdot\frac{\sqrt{n}}{n+1}$
5. $\sum_{n=2}^{\infty} (-1)^n\cdot\frac{2n+1}{3n-4}$
6. $\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}$
7. $\sum_{n=1}^{\infty} \frac{2}{n^4}$
8. $\sum_{n=0}^{\infty} 0.3\cdot 1.2^n$
9. $\sum_{n=0}^{\infty} \frac{3^n+1}{n!}$
10. $\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{2n+1}$
11. $\sum_{n=0}^{\infty} \frac{(-1)^n\cdot n!}{10^n}$
12. $\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{n^2}$
13. $\sum_{n=1}^{\infty} \frac{2-\cos n}{n^2}$
14. $\sum_{n=1}^{\infty} \frac{\sin n}{3^n}$
15. $\sum_{n=0}^{\infty} \frac{(-3)^n}{n!+n^2}$
16. Approximate the value of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\cdot 2^n}$ using s_{10}. Estimate how much error there is in this approximation.
17. Approximate the value of $\sum_{n=1}^{\infty} (-1)^n\cdot\frac{n}{n^3+1}$ using s_{15}. Estimate how much error there is in this approximation.
18. Approximate the value of $\sum_{n=1}^{\infty} \frac{n^2}{(-3)^{n+1}}$ using s_{20}. Estimate how much error there is in this approximation.
19. How many terms are needed to guarantee an approximation of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ that is within 0.05 of the actual value of the series?
20. How many terms are needed to guarantee an approximation of $\sum_{n=1}^{\infty} (-1)^n\cdot\frac{n}{n^3+10}$ that is within 0.05?
21. If one wants to compute the value of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ accurately to 3 decimal places, how many terms should be used?
22. If one wants to compute the value of $\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{n^2-1}$ accurately to 4 decimal places, how many terms should be used?
23. Use the 10th partial sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ and the alternating series error bound to give bounds on the value of the sum. That is, fill in the blanks: $\underline{\quad\quad} \le \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \le \underline{\quad\quad}$.
24. Use the 5th partial sum of $\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{(2n)!}$ and the alternating series error bound to give bounds on the value of the sum.
25. Look back at your answers to problems 16-24. Which appear to converge faster (i.e., have less error for an equal number of terms

in the partial sum), absolutely convergent or conditionally convergent series. Why?

26. An alternating series $\sum_{n=1}^{\infty} a_n$ has $|a_{n+1}| < |a_n|$ for all $n \geq 1$. Selected partial sums for the series are given as follows: $s_{100} = 3.42$, $s_{101} = 3.61$, and $s_{102} = 3.58$. Give bounds for the actual value of the series.

27. An alternating series $\sum_{n=1}^{\infty} a_n$ has $|a_{n+1}| < a_n$ for all $n \geq 1$. Selected partial sums for the series are given as follows: $s_{40} = 12.102$, $s_{41} = 11.956$, and $s_{42} = 12.089$. Give bounds for the actual value of the series

28. Use the fourth-order Maclaurin polynomial for $f(x) = \cos(x)$ to approximate cos(-1). Estimate the amount of error in your approximation using the alternating series error bound.

29. Based on the alternating series error bound, what degree Maclaurin polynomial is required to approximate sin(5) with error guaranteed to be less than 10^{-6}? How about sin(0.5)? How about sin(0.01)?

30. Recall that the Maclaurin polynomial of degree n for $f(x) = e^x$ is given by

$$P_n(x) = 1 + x + \tfrac{x^2}{2!} + \cdots + \tfrac{x^n}{n!}.$$

   a. Use $P_5(x)$ to approximate the value of e^{-2}.

   b. Use the alternating series error bound to estimate the amount of error in this approximation.

   c. Use Lagrange error bound to estimate the error in this approximation.

   d. In this particular case, which method is easier? Which gives a tighter bound on the error?

   e. Can you repeat parts (b)-(d) to estimate the error involved in approximating e^2 using $P_5(x)$? Why or why not?

31. Recall that the n^{th}-degree Maclaurin polynomial for $f(x) = \ln(1+x)$ is given by

$$P_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^n \cdot \frac{x^n}{n}.$$

   a. Use the fifth-degree Maclaurin polynomial to approximate $\ln(1.3)$.

   b. Use the alternating series error bound to provide bounds for the value of $\ln(1.3)$.

   c. Use Lagrange error bounds to estimate the error in your approximation from part (a). Which kind of error bound is more convenient to apply in this situation?

32. a. Use the fourth-degree Maclaurin polynomial for the cosine function to approximate cos(0.4).

   b. Use the alternating series error bound to provide bounds for the value of cos(0.4).

   c. Use Lagrange error bounds to estimate the error in your approximation from part (a). Which kind of error bound is more convenient to apply in this situation?

33. Recall that the n^{th}-degree Maclaurin polynomial for $f(x) = \arctan x$ is given by

$$P_n(x) = x - \tfrac{x^3}{3} + \tfrac{x^5}{5} - \cdots + (-1)^n \cdot \tfrac{x^{2n+1}}{2n+1}.$$

   a. Use the fact that $\pi = 4\arctan(1)$ and the third-degree Maclaurin polynomial for the arctangent function to approximate the value of π.

   b. Use the alternating series error bound to estimate the error in your approximation from part (a).

   c. How many terms would you need in order to obtain a decimal approximation of π that is guaranteed to be accurate to two decimal places? (Hint: Be careful of the 4.)*

* This approach to computing digits of π, while appealing in its simplicity, is completely useless in practice. Your answer to part (c) should indicate this. There are several series that converge to π much

34. Give an example of a series $\sum(-1)^n a_n$ (with $a_n > 0$) that converges absolutely, or explain why no such series exists.

35. Give an example of a series $\sum(-1)^n a_n$ (with $a_n > 0$) that diverges, or explain why no such series exists.

36. Give an example of a series $\sum a_n$ (with $a_n > 0$) that converges conditionally, or explain why no such series exists.

37. Give an example of a convergent series $\sum a_n$ such that $\sum a_n^2$ diverges.

38. Give two divergent series $\sum a_n$ and $\sum b_n$ such that $\sum(a_n + b_n)$ converges

In Problems 39-44, identify whether the statement is true or false. If it is true, give a proof or explanation of why. If it is false, give a counter-example or explanation.

39. If $\sum a_n^2$ converges, then $\sum |a_n|$ converges.

40. If $\sum a_n$ and $\sum -a_n$ both converge, then $\sum |a_n|$ converges.

41. If $\sum \frac{a_n}{n}$ converges then $\sum a_n$ converges.

42. If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

43. If $\sum a_n$ converges, then $\sum |a_n|$ converges.

44. If $\sum a_n$ converges absolutely, then $\sum(-1)^n a_n$ converges.

In Problems 45-59, find the radius <u>and</u> interval of convergence of the given power series.

45. $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n^2}$

46. $\sum_{n=0}^{\infty} \frac{n^2+3}{(2n)!} x^n$

47. $\sum_{n=0}^{\infty} \frac{(3x)^n}{2n+1}$

48. $\sum_{n=0}^{\infty} \frac{n!(x-6)^n}{n^2+5}$

49. $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$

50. $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}$

51. $\sum_{n=1}^{\infty} \frac{x^n}{n^2 \cdot 3^n}$

52. $\sum_{n=1}^{\infty} \frac{n!}{3^n + n^2} x^n$

53. $\sum_{n=1}^{\infty} \frac{(x-4)^n}{n^n}$

54. $\sum_{n=0}^{\infty} \left(\frac{x+1}{5}\right)^n$

55. $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

56. $\sum_{n=1}^{\infty} \frac{(x+2)^{2n}}{n^2+2n}$

57. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}(x+3)^n$

58. $\sum_{n=1}^{\infty} \frac{3^n}{2^n-1} x^n$

59. $\sum_{n=1}^{\infty} \frac{\sin n}{n^2} x^n$

60. Provide an example of a power series whose radius of converges is 0.

61. Provide an example of a power series whose radius of convergence is infinite.

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more rapidly. One improvement is to use *two* arctangents, for example $\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}$. Replacing these arctangents with their corresponding series produces an approximation that is accurate to all decimal places displayed by a TI-84 after only 13 terms. Question for you: Why does this approach converge so much more rapidly?

62. Provide an example of a power series whose radius of convergence is finite and which diverges at both endpoints of its interval of convergence.

63. Provide an example of a power series whose radius of convergence is finite and which converges at both endpoints of its interval of convergence.

64. Provide an example of a power series whose radius of convergence is finite and which converges at the right endpoint of its interval of convergence, but not at the left endpoint.

65. Let $f(x)=\sum_{n=0}^{\infty} x^n$.

    a. Find the interval of convergence of this power series.

    b. Express $f'(x)$ as a power series and find its interval of convergence.

    c. Express $\int_0^x f(t)dt$ as a power series and find its interval of convergence.

66. Let $f(x)=\sum_{n=1}^{\infty} \frac{(x-3)^n}{2n}$.

    a. Find the interval of convergence of this power series.

    b. Express $f'(x)$ as a power series and find its interval of convergence.

    c. Express $\int_3^x f(t)dt$ as a power series and find its interval of convergence.

67. Let $f(x)=\sum_{n=0}^{\infty} \frac{(-1)^n n^2}{3^n}(x+1)^n$.

    a. Find the interval of convergence of this power series.

    b. Express $f'(x)$ as a power series and find its interval of convergence.

    c. Express $\int_{-1}^x f(t)dt$ as a power series and find its interval of convergence.

68. The function *f* is defined by a power series as follows: $f(x)=\sum_{n=0}^{\infty} \frac{x^n}{2n+1}$.

    a. Find the interval of convergence of this series.

    b. Approximate $f(-1)$ by using a fourth-degree Maclaurin polynomial for *f*.

    c. Estimate the error in your approximation from part (a) and give bounds for the value of $f(-1)$.

69. The function *f* is defined by a power series as follows: $f(x)=\sum_{n=0}^{\infty} (-1)^n \cdot \frac{(x-3)^n}{2^n \cdot n!}$.

    a. Find the radius of convergence of this series.

    b. Use the third-degree Taylor polynomial for *f* centered at $x = 3$ to approximate $f(4)$.

    c. Estimate the error in your answer to part (b).

    d. How many terms are needed in a polynomial approximation of *f* to compute $f(4)$ with error less than 10^{-6}?

70. Given a power series $\sum c_n (x-a)^n$, suppose that $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L$, where L is positive and finite.

    a. What is the radius of convergence of this power series?

    b. Ignoring the question of endpoint convergence for the moment, what is the interval of convergence of this series? Give your answer in terms of *a* and *L*.

Section 9 – Conditional Convergence
(Optional)

In this section we will explore how conditionally convergent series can behave in ways that are just plain weird. The purpose of including this section is twofold. First, I hope that by seeing some of the strange things that happen in the realm of conditional convergence you will gain an understanding of why absolutely convergent series are "better" in some way. Second, some of the bizarre results you will see in this section are really wild, and I think that is interesting for its own sake. You may find that some of the ideas here push you to think a little more abstractly than you normally do, and that is good for your development as a mathematician.

Revisiting the Alternating Harmonic Series

We know by now that the alternating series,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots, \tag{9.1}$$

converges conditionally. I suggested in Section 8 that the sum of this series is ln(2), or about 0.6931. The argument was based on evaluating $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{x^n}{n}$, the power series for $\ln(1+x)$, at $x=1$. I don't know whether this will be hard to believe or not. But it really is a true statement. You can find it on the internet.

Another thing we know at this point in our mathematical education is that addition is commutative. The commutative property of addition basically says that when you add, the order in which you add the terms does not matter: $5+8=8+5$. Let's see how this plays out with the alternating harmonic series.

$$\begin{aligned}\ln 2&=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots\\&=\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\cdots\end{aligned}$$

All I have done is reorder the terms in a particular way. I started by grouping the 1 (the first positive term) with the 1/2 (the first negative term). Then I tacked on the next negative term that had not yet been used. Then I added another group, again consisting of the smallest positive and negative terms that had not yet been used. Then I tacked on the next remaining negative term. The process is repeated forever. Now that you have a sense for how this rearrangement is being performed, let's continue by simplifying the grouped expressions.

$$\begin{aligned}\ln 2&=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\cdots\\&=\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\cdots\\&=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\cdots\\&=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\right)\\&=\frac{1}{2}\ln 2\end{aligned}$$

We conclude that

$$\ln 2 = \tfrac{1}{2}\ln 2\,. \tag{9.2}$$

That certainly seems odd. I bet you thought the only number equal to half of itself was 0. I guess this means that $\ln 2 = 0$. But $\ln 1 = 0$. It appears then that 2 must equal 1. Or consider another path to this conclusion:

$\ln 2 = \tfrac{1}{2}\ln 2$	Equation (9.2)
$1 = \tfrac{1}{2}$	Divide by $\ln 2$.
$2 = 1$	Multiply through by 2.

We can "prove" a whole bunch of nonsense from Equation (9.2). For example, by the properties of logarithms this equation implies $\ln 2 = \ln 2^{1/2}$ or equivalently $2 = \sqrt{2}$. Hm. Strange.

Of course, if 2 = 1, then we can subtract 1 from both sides to obtain 1 = 0. At this point, it is not hard to show that all numbers are equal to zero and hence to one another. Mathematics as we know it is about to completely collapse. What went wrong?

The Non-Commutativity of Addition

Let's put our minds to rest about one truth. Equation (9.2) is not a valid statement. It is not true that $\ln 2$ is equal to half of itself. Our derivation of (9.2) was flawed, and the error came in the process of rearranging the terms of (9.1). There was no problem with the rearrangement per se; every term in (9.1) is used once and only once in the rearranged series. (You should take a moment to convince yourself that this statement is true. It is.) Rather the problem was in our expectation that rearranging the terms of (9.1) would not affect the value of the sum. It turns out that when there are infinitely many terms involved, addition is not necessarily commutative. The order in which you add the terms can affect the sum. To repeat: *addition is not always commutative when you are adding infinitely many things*. That ought to be surprising.*

We can be a little more specific about this new fact of mathematics. The series in (9.1) is <u>conditionally</u> convergent. The catastrophic events that unfolded in our derivation of (9.2) could never have happened if we had been working with an absolutely convergent series. In absolutely convergent series, the terms are heading to zero so quickly that sliding the terms around does not make any difference. In other words, addition in the absolutely convergent case follows the same rules of addition that we have always believed and treasured; addition is commutative for absolutely convergent series. Why are conditionally convergent series different? What is the defect in them that destroys the commutativity of addition?

One way to think about this is to recognize that any conditionally convergent series has infinitely many positive terms and infinitely many negative terms. If this were not the case, the series could not converge conditionally; it would either have diverged or converged absolutely. For example, if a series has finitely many negative terms, once you get past them you are left with what is effectively a positive-term series; if an exclusively positive-term series converges, it must do so absolutely (and the same for an exclusively negative-term series). Now imagine a conditionally convergent series (for example, (9.1)), and look at the collection of positive terms separately from the negative terms. In this way we can parse the series into two sub-series. If the series we started with was conditionally convergent, both of these

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* It is also a bit of a misstatement. Strictly speaking, addition is a binary operation; it acts on two numbers at a time. When dealing with only two numbers, it is certainly true that addition is commutative. From a technical standpoint, we shouldn't talk about commutativity with more than two summands. What I mean when I say that addition is not commutative is just that the order of the addition makes a difference to the sum. This is not quite the formal meaning of "commutative," but I hope you'll allow me to fudge a bit here.

sub-series must diverge. In the case of (9.1), the two sub-series are $\sum \frac{1}{2n+1}$ and $-\sum \frac{1}{2n}$, both of which clearly diverge by comparison to the harmonic series. This *always* happens with conditionally convergent series; it has to. If the sub-series did not both diverge to infinity, then they must converge. However, if the sub-series converge, they must do so because the terms are vanishing very quickly. (Remember that the terms in each sub-series all have the same sign, so the only way for them to converge is to do so absolutely—that is, quickly.) But if the terms go to zero quickly, then the original series should have converged absolutely, not conditionally. Therefore, we conclude the following Important Point:

Important Point: Any conditionally convergent series can be broken up into two divergent series, one consisting of only positive terms and the other of only negative terms.

If we let a_n stand for the terms of the original series, p_n be the positive terms, and q_n be the negative terms, then the Important Point says

$$\begin{aligned}\sum a_n &= \sum p_n + \sum q_n \\ &= \infty + (-\infty) \\ &= \infty - \infty.\end{aligned}$$

As we know, $\infty - \infty$ is an indeterminate form. Its value, if it has a one, is very sensitive to just how these infinities are approached. Every conditionally convergent series is a delicate balancing act of positive and negative terms. The precise balance is determined by the order in which the terms appear in the sum. Change the order, and you change the balance; the sum is affected.

Here is a different way to think about this. If a series Σa_n converges to a sum s, then that means that $\lim_{n\to\infty} s_n = s$. The sum of the series is simply the limit of the partial sums. But the partial sums depend very crucially on the order in which the terms are listed. If you scramble the terms, the sequence of partial sums will be a completely different sequence. There is no reason to expect that a *different* sequence of partial sums would converge to the same limit. From this perspective, it is actually surprising that an absolutely convergent series' value is *not* dependent on the order of the terms. This helps us appreciate that absolute convergence really is a more robust kind of convergence than conditional convergence.

Riemann's Rearrangement Theorem

We showed at the beginning of this section that the terms of the alternating harmonic series, which converges to $\ln 2$, can be rearranged to converge to half of that value. In fact, the situation is much worse. If we rearrange the terms just right, then we can make them converge to *any number we want*. The order we want to put the terms will depend on the target sum we are trying to hit.

For example, let's say we want to rearrange the terms of the alternating harmonic series in such a way as to force the rearranged series to converge to 1/3. This number is positive, so we will start with our positive terms, and we might as well start from the beginning with 1. But 1 is already too big. So let us start including the negative terms in our sum, in order, until our partial sum drops below 1/3.

$$\begin{aligned}1 &= 1 > \tfrac{1}{3} \\ 1 - \tfrac{1}{2} &= \tfrac{1}{2} > \tfrac{1}{3} \\ 1 - \tfrac{1}{2} - \tfrac{1}{4} &= \tfrac{1}{4} < \tfrac{1}{3}\end{aligned}$$

Now we have gone too far with the negative terms, so let's add positive terms until we overshoot 1/3 again.

$$\begin{aligned}1 - \tfrac{1}{2} - \tfrac{1}{4} &= \tfrac{1}{4} < \tfrac{1}{3} \\ 1 - \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{3} &= \tfrac{7}{12} > \tfrac{1}{3}\end{aligned}$$

We have overshot our goal again, so it is time to switch to negative terms. Skipping ahead a bit, we will get the following series:

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\frac{1}{7}-\frac{1}{14}-\frac{1}{16}+\frac{1}{9}-\frac{1}{18}-\frac{1}{20}+\frac{1}{11}-\frac{1}{22}-\frac{1}{24}+\cdots$$

This rearrangement of the alternating series will converge to one third.

If we had wanted the series to converge to π instead of 1/3, we could have done that as well. For any number you want, there is a rearrangement of the terms of the alternating harmonic series that gets there. And there is nothing special about the alternating harmonic series. What we have seen of the alternating harmonic series is true of all conditionally convergent series, and it is stated in a theorem due to Riemann (of Riemann sum fame).

Theorem 9.1 – Riemann's Rearrangement Theorem
For any conditionally convergent series and for any number s there exists a rearrangement of the terms of the series such that the new sequence of partial sums converges to s.

I think that's just crazy. It is also a reason why we as mathematicians feel a bit more comfortable with absolutely convergent series. Absolutely convergent series (like power series in the interior of their intervals of convergence) are simply much better behaved.

Section 10 – Taylor Series

Taking Stock of Where We Are

This is the final section of this chapter, so now would be a good time to look back at what our goals were from the beginning and see how we are doing so far. We closed Section 2 with four questions about modeling functions with polynomials. For reference, here they are.

1. *Is there a systematic way to come up with the coefficients of a Taylor polynomial for a given function?*
2. *Can we know how big the error will be from using a Taylor polynomial?*
3. *When can we extend the interval on which the Taylor polynomial is a good fit indefinitely?*
4. *Can we match a function perfectly if we use infinitely many terms? Would that be meaningful?*

We answered Question 1 in Section 3. The coefficient for the n^{th}-degree term should be $\frac{f^{(n)}(a)}{n!}$, where a is the center of the polynomial. Question 2 has been answered twice. In special cases, we can use the alternating series error bound from Section 8. The more general answer to this question, though, came in Section 4 in the form of Taylor's Theorem and the Lagrange error bound.

Questions 3 and 4 take us beyond polynomials and into the realm of power series. As you might expect, the answer to Question 3 has everything to do with interval of convergence, though we need to talk about Question 4 before we can appropriately frame Question 3 and its answer. In any event, the tools we developed in Sections 6-8 are crucial for determining intervals of convergence.

And finally we come to Question 4, the most important question for this chapter. The second part of this question—Would it be meaningful to have infinitely many terms?—has been answered already. A power series is a "polynomial" with infinitely many terms. Such a thing does indeed make sense, at least on its interval of convergence. As we saw at the end of Section 6 with the sine and natural logarithm functions, the answer to the first part—Can we match a function perfectly with a power series?—is: Yes! (Actually, we have to qualify that quite a bit. Most functions that we care about can be matched perfectly by a power series, though there are plenty of functions that can't be. And of course, even for the ones that can be represented as a power series, this representation only makes sense on the interval of convergence of the series.) The name we give a power series that represents a given function is a Taylor series (or a Maclaurin series if the center is 0).

Let's look at a couple examples, as we did in Section 6. We already know that the exponential function can be approximated by $P_n(x) = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n$. This is a Maclaurin *polynomial*. The Maclaurin *series* for the exponential function is simply $1 + x + \frac{1}{2!}x^2 \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$; just tack a "$+\cdots$" on the end. This power series is an exact representation of the exponential function. It is not an approximation. It <u>is</u> the function.*

For any power series, we must ask about the interval of convergence. We suspect from our work in Section 2 that the interval of convergence for the exponential series will be the entire real line. In fact, this is so, as the ratio test proves.

* In fact, we can take this power series to be the definition of the exponential function. In more advanced texts, having clear and precise definitions for transcendental functions is very important, but also somewhat complicated. One way to deal with this issue is to simply define $f(x) = e^x$ as the Maclaurin series we see here.

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 < 1$$

Since this limit is less than 1, independent of the *x*-value in question, the exponential series converges for all *x*. The radius of convergence is infinite.

But hang on a second. We know that the series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges for all *x*, but how do we know it converges *to the exponential function*? The partial sums (Maclaurin polynomials) were created to model the exponential function, but they were just models. How do we know that the series fits the function exactly for all *x*-values? We can prove that this is indeed the case by using Taylor's Theorem. Applied to the Maclaurin polynomial for the exponential function, Taylor's Theorem says that

$$\begin{aligned} e^x &= P_n(x) + R_n(x) \\ &= 1 + x + \tfrac{1}{2!}x^2 + \cdots + \tfrac{1}{n!}x^n + R_n(x) \end{aligned}$$

where $\left|R_n(x)\right| \le \frac{M}{(n+1)!}\left|x-0\right|^{n+1}$. In the case of the exponential function, we can take $M = 3^x$ if *x* is positive and $M = 1$ if *x* is negative (see Problem 24 of Section 4). This means that

$$\left|R_n(x)\right| \le \frac{\max\left(3^x, 1\right)}{(n+1)!}\cdot\left|x\right|^{n+1}. \tag{10.1}$$

To see what happens as we add infinitely many terms to the polynomial, we simply take the limit of (10.1).

$$\lim_{n\to\infty}\left|R_n(x)\right| \le \lim_{n\to\infty}\frac{\max\left(3^x,1\right)\cdot x^{n+1}}{(n+1)!} = 0$$

Whichever form *M* takes, it depends on *x*, not on *n*. As *n* goes to infinity, the factorial denominator will always dominate the numerator, driving the limit to 0. This tells us that the remainder—the error between the polynomial approximation and the actual value of the exponential function—goes to zero as we pass to the limiting case of the power series. For any *x*-value, the Maclaurin series for the exponential function has no error. This series <u>is</u> the function.

We will typically skip the verification that $R_n(x)$ goes to zero as $n \to \infty$, as it will be the norm for the functions that we care about. However, you should be aware that there are some naughty functions out there whose Taylor series converge, but do not converge to the function being modeled. You can play with some examples in the problems.

For our second example, let us look at the Maclaurin series for $f(x) = \ln(1-x)$. The Maclaurin polynomial for this function is given by $P_n(x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n}$, so we will brazenly assert that the Maclaurin *series* is

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} + \cdots = -\sum_{n=1}^{\infty}\frac{x^n}{n}.$$

Now that we have a power series, we need to know what the interval of convergence is. We apply the ratio test.

$$\lim_{n\to\infty}\left|\frac{-x^{n+1}}{n+1}\cdot\frac{n}{-x^n}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\cdot x\right| = \left|x\right|$$

We need $\left|x\right| < 1$, or $-1 < x < 1$. We now check for convergence at the endpoints.

$x = -1$: $\sum_{n=1}^{\infty} -\frac{(-1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$. This is the alternating harmonic series. It converges.

$x=1$: $\sum_{n=1}^{\infty} -\frac{1^n}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$. This is the (opposite of the) harmonic series. It diverges.

We conclude that the interval of convergence for this series is $-1 \le x < 1$. It's a good thing that the series diverges when x is 1. If we plug 1 in for x in $\ln(1-x)$, we obtain ln(0), which is undefined. The fact that our Maclaurin series is consistent with this feature of the function it represents lends credence to the idea that the series <u>is</u> the function, at least within the interval of convergence.

We are in danger of getting ahead of ourselves, so let's pause and state formally what we are talking about.

Taylor Series

> **Definition:** If a function *f* is differentiable infinitely many times in some interval around $x = a$, then the **Taylor series** centered at *a* for *f* is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.
> If $a = 0$, then we can call the series a **Maclaurin series**.

As you can see, a Taylor series is a special case of a power series where the coefficients c_n are determined by the function in question and the chosen center: $c_n = \frac{f^{(n)}(a)}{n!}$. Though it is not part of the definition, we have seen that the Taylor series for a function is an alternative representation of the function that (in the nice cases that we care about) is perfectly faithful to how we understand the function to behave. This is something that is still amazing to me; by analyzing the derivatives of *f* at a single point, we are able to build a representation of *f* that is valid anywhere within in the interval of convergence. For example, when we developed the Maclaurin series for the exponential function, we were able to create a power series that describes how the function behaves at *any* *x*-value by looking very closely at its behavior at $x = 0$. And we could just as easily have chosen a different center. By looking at, say, $x = 2$, we still would have extracted enough information to represent the function for all *x*. It is as if every point on the curve has all the information resting inside of it to create the entire function. The DNA (the values of the derivatives) is there in every cell (*x*-value) of the organism (the function). In the case of $f(x) = \ln(1-x)$, the power series even seemed to know where the vertical asymptote would be! The interval of convergence ended just as we hit the edge of the domain of *f*. How did it know?

In practice, we often do not need to resort to the definition to build a Taylor series for a given function. First of all, so many of the functions we care about have been done already; the table at the end of Section 3 can be extrapolated with a "$+\cdots$" to give Maclaurin series for the most commonly encountered functions. It remains for us, of course, to find their intervals of convergence.

Furthermore, we can use the same tricks that we used to generate new Taylor polynomials to generate new Taylor series. We can substitute into known Taylor series, manipulate known series algebraically, differentiate and integrate term by term, and exploit the geometric series sum formula just as we have been doing all along. It is only when we encounter a legitimately new function or are changing the center of the series expansion that we have to go back to the definition for guidance in determining the coefficients.

Another Perspective on Radius of Convergence

The standard way to determine radius of convergence is to apply the ratio test to the absolute value of the general term of the series. However, there is a short-cut. *The radius of convergence of a Taylor* *series is always as big as it can be.* Look back at the example of the Maclaurin series for $f(x) = \ln(1-x)$. This series could not possibly converge at $x = 1$ since the function it represents is not defined there. But the series *does* converge for every x-value between 1 and the center. The radius of convergence is 1—the distance from the center to the end of the function's domain. On the other side, the interval of convergence ended at $x = -1$ because the Maclaurin series must have an interval of convergence that is symmetric (up to the endpoints) about its center.

This will always be the case. The radius of convergence will always be as large as it can be without including any singularities of the function. Consider $g(x) = \dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. This series is geometric, with $r = x$, so it converges for $-1 < x < 1$. This is as big an interval of convergence as you can have with a center at $x = 0$. The vertical asymptote in the graph of g is at $x = 1$, one unit away from the center, so the radius of convergence is 1. If you choose a different center, you can find a Taylor series with a different radius of convergence because the distance from the center to the asymptote will be different. This idea is explored in the problems.

But what about $h(x) = \dfrac{1}{1+x^2}$? The domain of h includes all real numbers. Yet the Maclaurin series $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ is a geometric series with $r = -x^2$, so it converges only for $-1 < x < 1$. This seems to contradict the point I am trying to make about an interval of convergence being as big as it can be before including a singularity of the function. We have to look in a different direction to understand this function, this time into the complex plane. The function h is defined for all *real* numbers, but it is not defined for all numbers. Specifically, h is undefined if $x = \pm i$, as either of these x-values makes the denominator vanish. Both i and $-i$ are one unit away from the origin of the complex plane, so the radius of convergence of the Maclaurin series for h can be no larger than 1. In fact, the Maclaurin series for h converges for *every* number—real and complex—that is within one unit of the origin in the complex plane.

Suddenly all the terminology involving centers and radii of convergence make more sense. These terms suggest circles, and you may have been wondering where these circles are. They are in the complex plane. A power series converges for every complex number z that is within R units of the center (R being the radius of convergence). We do not need to go into the complex plane much for this course, but it does make sense of the terms that we have been using all along.

Applications of Taylor Series: Integrals, Limits, and Sums

Okay, so why do we care? I care about Taylor series because I think they are really, really cool. You are free to disagree, but if you do I will have to present some more concrete examples of their utility.

The first has to do with filling a major hole in our ability to do calculus. Your calculus course has been mainly about finding and using derivatives and antiderivatives. Derivatives are no problem. Between the definition of the derivative and the various differentiation rules, you should be able to quickly write down the derivative of any differentiable function you meet. Antidifferentiation is trickier; many functions simply do not have explicit antiderivatives. We cannot integrate them. Taylor series provide us with a way for circumventing this obstacle, at least to a certain extent.

Example 1
Evaluate $\int e^{x^2}\,dx$.

Solution

The function $f(x)=e^{x^2}$ has no explicit antiderivative, so we cannot approach this problem directly. However, we can replace e^{x^2} with its Maclaurin series. Recall that

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

whence it follows by substitution that

$$\begin{aligned} e^{x^2} &= 1 + \left(x^2\right) + \frac{\left(x^2\right)^2}{2!} + \cdots + \frac{\left(x^2\right)^n}{n!} + \cdots \\ &= 1 + x^2 + \frac{x^4}{2!} + \cdots + \frac{x^{2n}}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}. \end{aligned}$$

We can integrate this series term by term.

$$\begin{aligned} \int e^{x^2}\,dx &= \int \left(1 + x^2 + \frac{x^4}{2!} + \cdots + \frac{x^{2n}}{n!} + \cdots \right) dx \\ &= C + x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} + \cdots + \frac{x^{2n+1}}{(2n+1)\cdot n!} + \cdots \end{aligned}$$

I put the constant of integration at the front to avoid confusion with the ellipsis at the end.

In one sense, this is only a partial solution to the problem. We still cannot get an explicit antiderivative, but we can at least give a series representation: $\int e^{x^2}\,dx = C + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\cdot n!}$. As needed, partial sums can be taken for computation. ◊

This idea is also useful for definite integrals.

Example 2

Compute $\int_0^1 e^{x^2}\,dx$.

Solution

$$\begin{aligned} \int_0^1 e^{x^2}\,dx &= \left[x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} + \frac{x^7}{7 \cdot 4!} + \cdots \right]_0^1 \\ &= 1 + \frac{1}{3} + \frac{1}{5 \cdot 2!} + \frac{1}{7 \cdot 4!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)\cdot n!} \end{aligned}$$

Again, partial sums can be used for specific computation. ◊

Practice 1

Use Taylor series to compute $\int \sin\left(x^3\right) dx$ and $\int_0^{1/2} \sqrt{1+x^4}\,dx$.

Another handy application of Taylor series is to simplify the computation of limits. We saw a glimmer of this back in Section 2, long before we were ready to really make use of it. But now that we know that a Taylor series for a function <u>is</u> the function, we are justified in replacing any function with its Taylor series.

Example 3

Use Maclaurin series to help evaluate $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$ and $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}$.

Solution

We replace $\sin x$ with its Maclaurin series in the first limit.

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots}{x}=\lim_{x\to 0}\left(1-\tfrac{x^2}{3!}+\tfrac{x^4}{5!}-\cdots\right)$$

Every term other than the first has an x in it, and will therefore go to zero as $x\to 0$. This leaves only the leading 1 remaining. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$.

We use the same trick with the second limit, replacing $\cos x$ with its Maclaurin series.

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)}{x}=\lim_{x\to 0}\frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\cdots}{x}=\lim_{x\to 0}\left(\frac{x}{2!}-\frac{x^3}{4!}+\frac{x^5}{6!}-\cdots\right)=0$$

This time all the terms have an x in them after simplifying, so the limit reduces quickly and easily to 0.* ◊

Practice 2

Use a Maclaurin series to help evaluate $\lim\limits_{x\to 0}\dfrac{1-e^x}{x^2}$.

We can even sometimes use Taylor series to sneak out a way to find the sum of a thorny infinite series of constants. Despite spending all this time testing whether series converge, we have not had much success finding the actual sum of a series unless it is geometric or telescoping. We have evaluated a couple other sums here and there by using special tricks, and Taylor series provide yet another way.

Example 4

Evaluate the following series.

a. $1-\dfrac{1}{2!}+\dfrac{1}{4!}-\dfrac{1}{6!}+\cdots$ b. $\tan\frac{1}{2}-\dfrac{(\tan\frac{1}{2})^3}{3}+\dfrac{(\tan\frac{1}{2})^5}{5}-\dfrac{(\tan\frac{1}{2})^7}{7}+\cdots$

Solution

a. The general term of this series is $(-1)^n\cdot\dfrac{1}{(2n)!}$. The series alternates, and the denominators are the factorials of even numbers. That reminds us of the cosine Maclaurin series whose general term is $(-1)^n\cdot\dfrac{x^{2n}}{(2n)!}$. In fact, this is the given series, except that x^{2n} has become 1. This can easily happen if $x=1$. Hence the given series is the cosine series evaluated at x = 1. Its value must be cos(1). I encourage you to compute some partial sums to check.

(Note: Another solution to the equation $x^{2n}=1$ is $x=-1$. Was it right of me to ignore this possibility?)

* These limits are easily evaluated using l'Hôpital's Rule, but that is generally frowned upon. You may recall that these limits were important in developing the derivatives of the sine and cosine functions, so using l'Hôpital's Rule—which assumes you already know the derivatives—is circular. However, if we define these functions as their power series, as mentioned in an earlier footnote, the limits become a cinch. On the other hand, since we can also differentiate a power series term by term, we no longer need these limits to find the derivatives…

b. There are a couple clues here, the most obvious of which is the bizarre, recurring appearance of $\tan\frac{1}{2}$ in the series. This already puts me in mind of tangents or perhaps arctangents. In any event, the general term of the series is $(-1)^n \cdot \frac{(\tan\frac{1}{2})^{2n+1}}{2n+1}$. The series is alternating, has only odd powers, and has only odd denominators—not factorial denominators. A quick scan of the table of Maclaurin polynomials in Section 3 shows that this is indeed consistent with the arctangent series whose general term is $(-1)^n \cdot \frac{x^{2n+1}}{2n+1}$. This series is the arctangent series evaluated at $x = \tan\frac{1}{2}$; its value is $\arctan\left(\tan\frac{1}{2}\right) = \frac{1}{2}$. ◊

Practice 3
Evaluate the following very important sums.

a. $1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$ b. $1-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots$

In the problems you will have the opportunity to explore some other crafty approaches to using series for evaluating sums.

Applications of Taylor Series: Differential Equations

Another important use of Taylor series is in the solution of differential equations. Since many phenomena in the physical sciences are expressed as differential equations (Newton's Third Law, for example, states that $F = m \cdot \frac{d^2x}{dt^2}$.), solving differential equations is an indispensable tool for understanding the world around us. However, it is frequently the case that the differential equations we use as models are unsolvable. Taylor series give us a way to handle such situations.

There are two ways that a differential equation can be unsolvable to us. The first is if its solution requires an integration that we cannot perform. An example of this would be the differential equation

$$\frac{dy}{dx} = \sin\left(x^2\right).$$

But this difficulty is not new to us; overcoming it amounts to no more than using Taylor series for antidifferentiation, a topic that we just discussed.

The other way that a differential equation can stymie us is if it is not separable. This is where Taylor series come into play in a novel and interesting way.

Example 5
Solve the initial value problem $y' = x - 2y$, $y(0) = 0$.

Solution

First, convince yourself that this differential equation is not separable and therefore cannot be solved by methods you have probably learned so far.

We will begin by supposing that there is a solution of the form

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots. \qquad (10.2)$$

That is, we assume that there is a Maclaurin series for *y*. We also need y' since it appears in the differential equation. That is obtained simply by differentiating.

$$y' = c_1 + 2c_2x + 3c_3x^2 + \cdots \qquad (10.3)$$

From (10.2) it follows that $x - 2y$ (the right side of the differential equation) is given by

$$x-2\left(c_0+c_1x+c_2x^2+c_3x^3+\cdots\right)=-2c_0+\left(1-2c_1\right)x-2c_2x^2-2c_3x^3-\cdots. \tag{10.4}$$

Combining (10.3) and (10.4), we obtain the following.

$$y'=x-2y$$
$$c_1+2c_2x+3c_3x^2+4c_4x^3+\cdots=-2c_0+\left(1-2c_1\right)x-2c_2x^2-2c_3x^3-\cdots$$

Now we equate the coefficients of like powered terms. For example, the constant term on the left side must equal the constant term on the right. Therefore

$$c_1=-2c_0.$$

Continuing on in this fashion we obtain the following relationships.

$$\begin{aligned}2c_2&=1-2c_1\\3c_3&=-2c_2\\4c_4&=-2c_3\\&\vdots\end{aligned}$$

In general, for $n>2$, $nc_n=-2c_{n-1}$.

Now this is a big mess of coefficients, but we can use our initial condition to compute as many of them as desired. Since $y(0)=0$, Equation (10.2) becomes

$$0=c_0+c_1\cdot 0+c_2\cdot 0^2+c_3\cdot 0^3+\cdots.$$

Therefore, $c_0=0$. Since $c_1=-2c_0$, $c_1=0$ as well. We can continue to just plug and chug.

$$\begin{aligned}2c_2&=1-2\cdot c_1\Rightarrow 2c_2=1-2\cdot 0\Rightarrow c_2=\tfrac{1}{2}\\3c_3&=-2c_2\Rightarrow 3c_3=-2\cdot\tfrac{1}{2}=-1\Rightarrow c_3=\tfrac{-1}{3}\\4c_4&=-2c_3\Rightarrow 4c_4=-2\cdot\tfrac{-1}{3}=\tfrac{2}{3}\Rightarrow c_4=\tfrac{1}{6}\\5c_5&=-2c_4\Rightarrow 5c_5=-2\cdot\tfrac{1}{6}=\tfrac{-1}{3}\Rightarrow c_5=\tfrac{-1}{15}\\&\vdots\end{aligned}$$

Hence, the solution to our initial value problem is $y=\frac{1}{2}x^2-\frac{1}{3}x^3+\frac{1}{6}x^4-\frac{1}{15}x^5+\cdots$.

We have our answer, and it is fine to consider the problem solved at this point. However, I would like to look closer at the solution. Had we not reduced any of the fractions in computing the c_ns, we would have obtained

$$\begin{aligned}y&=\tfrac{1}{2}x^2-\tfrac{2}{6}x^3+\tfrac{4}{24}x^4-\tfrac{8}{120}x^5+\cdots\\&=\tfrac{1}{4}\left(2x^2-\tfrac{8x^3}{6}+\tfrac{16x^4}{24}-\tfrac{32x^5}{120}+\cdots\right)\\&=\tfrac{1}{4}\left(\tfrac{(2x)^2}{2}-\tfrac{(2x)^3}{6}+\tfrac{(2x)^4}{24}-\tfrac{(2x)^5}{120}+\cdots\right)\\&=\tfrac{1}{4}\left(\tfrac{(2x)^2}{2!}-\tfrac{(2x)^3}{3!}+\tfrac{(2x)^4}{4!}-\tfrac{(2x)^5}{5!}+\cdots\right).\end{aligned}$$

Pulling out the 1/4 was a trick to make the powers of 2 in the numerators match the powers on the *x*s that were already there. When we rewrite the series this way, it looks a lot like some kind of exponential because of the factorial denominators. However, a $2x$ has been substituted for x, and the series alternates, which suggests that the $2x$ is actually a $-2x$.

$$y=\tfrac{1}{4}\left(\tfrac{(-2x)^2}{2!}+\tfrac{(-2x)^3}{3!}+\tfrac{(-2x)^4}{4!}+\tfrac{(-2x)^5}{5!}+\cdots\right)$$

We are just missing the first two terms in the series for e^{-2x}, namely 1 and $-2x$. Therefore, I will take the leap and suggest that

$$y=\tfrac{1}{4}\left(e^{-2x}-(1-2x)\right)=\tfrac{1}{4}\left(2x-1+e^{-2x}\right).$$

I leave it to the reader to verify that this function does indeed satisfy both the differential equation and the initial condition. ◊

The fact that the solution to Example 5 was actually an elementary function begs the question of whether we could have found it by some direct method. As it turns out, there is a method for solving differential equations like the one in Example 5 without resorting to series. The trick is to use something called an integrating factor, and you will learn about it if you take a course in differential equations. In fact, you will learn several tricks for solving differential equations, but none of them is a cure-all. There are still plenty of differential equations that cannot be solved except by series expansion. We consider one in our next example.

Example 6

The Airy equation* is the differential equation $y'' - xy = 0$ or $y'' = xy$. Solve this equation subject to the initial conditions $y(0) = 1$ and $y'(0) = 0$.

Solution

The general approach is just like Example 5, just with an extra derivative. We again suppose that there is a Maclaurin series solution of the form

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots. \tag{10.5}$$

It follows that

$$y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 \cdots \tag{10.6}$$

and

$$y'' = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \cdots. \tag{10.7}$$

Also, using (10.5) we can determine that

$$xy = c_0x + c_1x^2 + c_2x^3 + c_3x^4 + c_4x^5 + c_5x^6 + \cdots. \tag{10.8}$$

Now we can represent the differential equation by combining (10.7) and (10.8).

$$2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \cdots = c_0x + c_1x^2 + c_2x^3 + c_3x^4 + c_4x^5 + c_5x^6 + \cdots \tag{10.9}$$

Equating the coefficients of like powers of *x* from (10.9) gives the following equalities.

$$\begin{aligned} 2c_2 &= 0 \\ 6c_3 &= c_0 \\ 12c_4 &= c_1 \\ 20c_5 &= c_2 \\ 30c_6 &= c_3 \\ &\vdots \\ n(n-1)c_n &= c_{n-3} \\ &\vdots \end{aligned}$$

We see immediately that $c_2 = 0$. But since $20c_5 = c_2$, we must have $c_5 = 0$ as well. In fact, since every coefficient is a multiple of the coefficient that came three before it, we must have $c_8 = 0, c_{11} = 0$, and in general $c_{3k+2} = 0$ where *k* is any non-negative integer.

Now it is time to use our initial conditions. Let's start with $y'(0) = 0$. From Equation (10.6), this condition implies that $c_1 = 0$. In turn, this means that $c_4 = 0$, $c_7 = 0$, and in general that $c_{3k+1} = 0$.

The condition that $y(0) = 1$, when substituted into (10.5), implies that $c_0 = 1$. This in turn implies that $6c_3 = 1$, or $c_3 = \frac{1}{6}$. Next, $30c_6 = \frac{1}{6}$, so $c_6 = \frac{1}{180}$. The particular solution to this differential equation is

* The Airy equation originally came up in George Airy's studies of the rainbow and has since been useful in quantum mechanics and other areas. The particular solution to the Airy equation that we will look at in this example is significant in that it has a turning point—a point where the function switches from being oscillatory like a sine or cosine to explosive like an exponential.

$$y = 1 + \tfrac{1}{6}x^3 + \tfrac{1}{180}x^6 + \cdots. \tag{10.10}$$

To my knowledge, there is no explicit formula for this function. Some partial sums (i.e., Maclaurin polynomials) for this function are shown in Figure 10.2. The black curve is the 99th partial sum. ◊

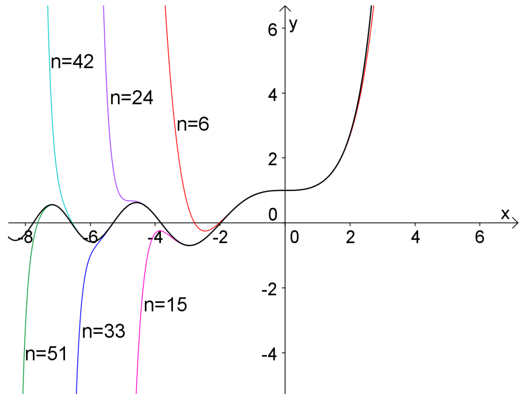

Figure 10.1: Maclaurin polynomials of degree *n* for the Airy function (10.10)

Euler's Formula

As a final example of the usefulness and beauty of Taylor series, we will derive an amazing and surprising result called Euler's Formula. This will involve examining functions whose inputs are allowed to be complex numbers. We have not looked at this yet, except briefly in our discussion of radius of convergence, but there is nothing special that we will need to worry about for what we are about to do. The number i is a constant, and it acts just like any other constant. It is useful to recall the powers of i before we begin.

$$\begin{gathered} i^0 = 1 \\ i^1 = i \\ i^2 = -1 \\ i^3 = i^2 \cdot i = -i \\ i^4 = i^3 \cdot i = 1 \\ i^5 = i^4 \cdot i = i \\ \vdots \end{gathered}$$

Notice that the values repeat cyclically.

We begin by considering the function e^{ix} as a Maclaurin series. To do this, we simply need the regular exponential series with (ix) substituted for x.

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots \\ &= 1 + ix + \frac{-x^2}{2!} + \frac{-ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \frac{-x^6}{6!} + \frac{-ix^7}{7!} + \cdots \end{aligned}$$

In the second line all that we have done is simplify the powers of i.

Next we will group the real terms and the imaginary terms, factoring the i out of the imaginary group. This regrouping is legal because the series for the exponential function converges absolutely for all inputs.

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{-x^2}{2!} + \frac{-ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \frac{-x^6}{6!} + \frac{-ix^7}{7!} + \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \end{aligned}$$

The two grouped series should look familiar. They are simply the cosine and sine series respectively. We conclude from this that

$$e^{ix} = \cos x + i\sin x .$$

This is Euler's Formula.

Observe the crazy things that Euler's Formula suggests. It tells us that when we look in the complex plane there is a certain equivalence between sinusoidal and exponential functions. The exponential function suddenly has a periodic or oscillatory behavior. Perhaps even stranger, the sine and cosine functions have exponential character; they grow without bound in the complex plane!

Euler's Formula is not just a curious result of manipulating Maclaurin series. It is tremendously useful for computation with complex numbers. We will show just a hint of this by plugging in π for x.

$$e^{i\pi} = \cos\pi + i\sin\pi$$

$$e^{i\pi} = -1 + i\cdot 0$$

$$e^{i\pi} = -1$$

This is absolutely insane. The number e is a transcendental number, a number that is impossible to express precisely in our decimal number system. We take that number, raise it to the imaginary transcendental number $i\pi$, and we get an integer! It is almost unbelievable. Not only do we get an integer, but we get a *negative* integer. Euler's formula shows us that exponential quantities can be negative when their inputs are complex, something that cannot happen for real inputs.

Mathematicians often like to take our last result and add 1 to both sides, giving

$$e^{i\pi} + 1 = 0 .$$

This relatively simple formula relates the five most important constants in mathematics: 0 (the additive identity), 1 (the multiplicative identity), π (the ratio of the circumference to the diameter of any circle), e (the base of the natural logarithm), and i (the imaginary constant). It is a truly remarkable statement that ties together many areas of mathematics that we otherwise might never have imagined to be connected. Basic arithmetic, the numbers e and π, the trigonometric functions, the exponential function, and complex numbers all arose at different points in history to solve completely different problems. But Euler's Formula shows us that for some reason these ideas and tools were all inevitably related from the start. There is something deep going on here, and if that is not an invitation to continue studying mathematics, I am not sure what is.

Answers to Practice Problems

1. We start with finding a Taylor series for the first integrand by substituting into the Maclaurin series for the sine function.

$$\begin{aligned} \sin\left(x^3\right) &= x^3 - \frac{\left(x^3\right)^3}{3!} + \frac{\left(x^3\right)^5}{5!} - \cdots + (-1)^n \cdot \frac{\left(x^3\right)^{2n+1}}{(2n+1)!} + \cdots \\ &= x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots + (-1)^n \cdot \frac{x^{6n+3}}{(2n+1)!} + \cdots \end{aligned}$$

Now for the integral.

$$\int \sin\left(x^3\right)dx = \int\left(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \cdots\right)dx$$
$$= C + \frac{x^4}{4} - \frac{x^{10}}{10\cdot 3!} + \frac{x^{16}}{16\cdot 5!} - \frac{x^{22}}{22\cdot 7!} + \cdots$$
$$= C + \sum_{n=0}^{\infty}(-1)^n \cdot \frac{x^{6n+4}}{(6n+4)\cdot(2n+1)!}$$

For the second integral, we do not have a stock series to use for the integrand, so we will create one from scratch. To avoid lots of chain and quotient rule, though, we will find a series for $f(x) = \sqrt{1+x}$ and then substitute into that later. The computations are simplest if we use a Maclaurin series.

$$\begin{array}{lcccc}
f(x) = (1+x)^{1/2} & \to & f(0) = 1 & \to & \dfrac{1}{0!} \\
f'(x) = \frac{1}{2}(1+x)^{-1/2} & \to & f'(0) = \frac{1}{2} & \to & \dfrac{1}{2\cdot 1!} \\
f''(x) = \frac{-1}{4}(1+x)^{-3/2} & \to & f''(0) = \frac{-1}{4} & \to & \dfrac{-1}{4\cdot 2!} \\
f'''(x) = \frac{3}{8}(1+x)^{-5/2} & \to & f'''(0) = \frac{3}{8} & \to & \dfrac{3}{8\cdot 3!} \\
f^{(4)}(x) = \frac{-15}{16}(1+x)^{-7/2} & \to & f^{(4)}(0) = \frac{-15}{16} & \to & \dfrac{-15}{16\cdot 4!} \\
f^{(5)}(x) = \frac{105}{32}(1+x)^{-9/2} & \to & f^{(5)}(0) = \frac{105}{32} & \to & \dfrac{105}{32\cdot 5!}
\end{array}$$

Using this tableau we find that

$$f(x) = 1 + \frac{1}{2\cdot 1!}x - \frac{1}{4\cdot 2!}x^2 + \frac{3}{8\cdot 3!}x^3 - \frac{15}{16\cdot 4!}x^4 + \cdots = 1 + \sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{(2n-3)!!\cdot x^n}{2^n\cdot n!}.$$

Substituting gives

$$\sqrt{1+x^4} = 1 + \frac{1}{2}x^4 - \frac{1}{4\cdot 2!}x^8 + \frac{3}{8\cdot 3!}x^{12} - \cdots = 1 + \sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{(2n-3)!!\cdot x^{4n}}{2^n\cdot n!}.$$

And now we are ready to integrate.

$$\int_0^{1/2}\sqrt{1+x^4}\,dx = \int_0^{1/2}\left(1 + \frac{1}{2}x^4 - \frac{1}{4\cdot 2!}x^8 + \frac{3}{8\cdot 3!}x^{12} - \cdots\right)dx$$
$$= \left[x + \frac{x^5}{5\cdot 2} - \frac{x^9}{9\cdot 4\cdot 2!} + \frac{3x^{13}}{13\cdot 8\cdot 3!} - \cdots\right]_0^{1/2}$$
$$= \frac{1}{2} + \frac{(1/2)^5}{5\cdot 2} - \frac{(1/2)^9}{9\cdot 4\cdot 2!} + \frac{3(1/2)^{13}}{13\cdot 8\cdot 3!} - \cdots$$
$$= \frac{1}{2} + \frac{1}{2^5\cdot 5\cdot 2} - \frac{1}{2^9\cdot 9\cdot 4\cdot 2!} + \frac{3}{2^{13}\cdot 13\cdot 8\cdot 3!} - \cdots$$
$$= \frac{1}{2} + \frac{1}{2^6\cdot 5} - \frac{1}{2^{11}\cdot 9\cdot 2!} + \frac{3}{2^{16}\cdot 13\cdot 3!} - \cdots$$
$$= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n-3)!!}{2^{5n+1}\cdot(4n+1)\cdot n!}$$

This can be approximated to any desired accuracy by using a partial sum. Better yet, the series is alternating which means that you can even estimate the error fairly easily.

2. $\lim_{x\to 0}\frac{1-e^x}{x^2}=\lim_{x\to 0}\frac{1-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)}{x^2}=\lim_{x\to 0}\frac{-x-\frac{x^2}{2!}-\frac{x^3}{3!}-\cdots}{x^2}=\lim_{x\to 0}\left(\frac{-1}{x}-\frac{1}{2!}-\frac{x}{3!}-\cdots\right)$

Because of the 1/*x*, this limit does not exist.

3. The series in part (a) shows every factorial in the denominator, and this suggests the series for e^x. In fact, it is this series, but with 1 substituted for *x*. Hence the value of this series is simply *e*.

In part (b), we have the same series, but it alternates. Alternation can be introduced by substituting a negative *x*-value, and this series sums to e^{-1} or 1/*e*.

Section 10 Problems

For all problems in which you are asked to find a Taylor series, you should include at least three non-zero terms and the general term of the series unless otherwise noted.

1. Find the Maclaurin series for $f(x) = \cos x$ and its interval of convergence.
2. Use Taylor's Theorem and the Lagrange error bound to show that the Maclaurin series for $f(x) = \cos x$ converges to the cosine function.
3. Find the Maclaurin series for $f(x) = \dfrac{1}{1-x}$ and its interval of convergence.
4. Find the Maclaurin series for $f(x) = \dfrac{1}{1+x^2}$ and its interval of convergence.
5. Find the Maclaurin series for $f(x) = \arctan x$ and its interval of convergence.
6. Find the Taylor series for the $f(x) = \ln(x)$ centered at $x = 1$. Also find the interval of convergence.
7. Find the Taylor series for $f(x) = \cos(x)$ centered at $x = \frac{-4\pi}{3}$. No general term is required.
8. Find the Taylor series for $f(x) = e^x$ centered at $x = e$.
9. Find the Taylor series for $f(x) = \sin(x-2)$ centered at $x = 2$. (Hint: This problem is easier than it might appear.)
10. Find the Maclaurin series for $f(x) = \arctan\left(x^2\right)$ and its interval of convergence.
11. Find the Maclaurin series for $f(x) = xe^x$.
12. Find the Maclaurin series for $f(x) = 2xe^{x^2}$ three ways (see next column).
    a. Use the definition of Taylor series to find the appropriate coefficients.
    b. Substitute into the series for e^x and then multiply through by $2x$.
    c. Differentiate the series for $g(x) = e^{x^2}$.
    d. Do your answers from parts (a), (b), and (c) agree?
13. Find the Maclaurin series for $f(x) = \cos^2 x$. (Hint: $2\cos^2 x = 1 + \cos(2x)$.)
14. Find the Maclaurin series for $f(x) = 2x^3 \sin x \cos x$.
15. Find the Taylor series for $f(x) = \sqrt[3]{x}$ centered at x = 8. No general term is required. However, make a conjecture of the radius of convergence of this series. (Hint: Look at a graph of *f* along with a few partial sums of the Taylor series.)
16. Find the Maclaurin series for $\ln(4+x)$ centered at x = -3 and its interval of convergence.
17. The **binomial series** is the Maclaurin series expansion for the function $f(x) = (1+x)^k$, where *k* is a constant.
    a. Show that the binomial series is
    $$1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots.$$
    b. Show that the series from part (a) terminates if *k* is a positive integer. In this case, the binomial series agrees with the binomial theorem.
    c. Assuming that *k* is not a positive integer, find the radius of convergence of the binomial series.
    d. Use the binomial series to find Maclaurin series for the following functions.
        i. $g(x) = (1+x)^{5/2}$
        ii. $h(x) = \dfrac{2}{\left(1+x^2\right)^3}$

iii. $k(x) = \dfrac{1}{\sqrt{1-x^2}}$

e. Use your series for $k(x)$ to find the Maclaurin series for $l(x) = \arccos(x)$. No general term is required.

18. Use the binomial series to explain why $(x+1)^{1/m} \approx 1 + \frac{1}{m}x + \frac{1-m}{2m^2}x^2$ when *x* is small in absolute value.

19. The derivatives at 0 of a function *f* are given by $f^{(n)}(0) = \dfrac{(-1)^n \cdot (n+1)}{n^2}$ for $n > 0$. Furthermore, $f(0) = 3$. Find the Maclaurin series for *f* and determine its interval of convergence.

20. The derivatives at 0 of a function *g* are given by $g^{(n)}(0) = \dfrac{3^n + n^2}{2^n}$ for $n \geq 0$. Find the Maclaurin series for g and determine its interval of convergence.

21. The derivatives at $x = 2$ of a function *h* are given by $h^{(n)}(2) = \dfrac{n!}{n+1}$ for $n \geq 0$. Find the Taylor series for *h* centered at $x = 2$ and determine its interval of convergence.

22. The derivatives at $x = -1$ of a function *f* are given by $f^{(n)}(-1) = (-1)^n \cdot \dfrac{n!}{2^n + n^2}$ for $n \geq 2$. $f(-1) = 8$ and $f'(-1) = 0$.

    a. Find the Taylor series for *f* centered at $x = -1$. Determine its interval of convergence.

    b. Does *f* have a local maximum, local minimum, or neither at $x = -1$?

    c. Find the Maclaurin series for $g(x) = f(x-1)$ and for $h(x) = g\left(x^2\right)$.

23. The Maclaurin series for a function *f* is given by $f(x) = \sum_{n=0}^{\infty} \dfrac{2^n \cdot (n+1)}{n!} x^n$. What is the value of $f^{(4)}(0)$? Can you determine the value of $f^{(4)}(1)$?

24. The Taylor series centered at $x = 3$ for a function *g* is given by $g(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n \cdot 3^n}{n^2+1}(x-3)^n$. Evaluate $g^{(5)}(3)$.

25. What is the coefficient of the 12th-degree term of the Maclaurin polynomial for $f(x) = \cos\left(x^3\right)$?

26. What is the coefficient of the sixth-degree term of the Maclaurin polynomial for $g(x) = x^2 e^x$?

27. Find the Maclaurin series for $f(x) = \dfrac{x}{1-x-x^2}$. You can either do this by using the definition of a Maclaurin series, or you can cross multiply and determine the coefficients in a manner similar to that of Examples 5 and 6.

28. Let $f(x) = \ln\left(\frac{1+x}{1-x}\right)$.

    a. Find the Maclaurin series for *f*.

    b. Find the interval of convergence for your series from part (a).

    c. Even though the interval of convergence for this series is not the entire real line, the series from part (a) is quite powerful. Show that any positive number α can be written as $\frac{1+x}{1-x}$ for some *x* in the interval of convergence.

    d. As a specific example, determine the value of *x* that corresponds to $\alpha = 15$. Use the seventh-degree Maclaurin polynomial for *f* to estimate ln(15).

29. The hyperbolic functions $\sinh(x)$ and $\cosh(x)$ are defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

and

$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$

a. Find Maclaurin series for both these functions. Also determine their intervals of convergence. (Hint: You can do this either combining exponential series or by determining the derivatives of hyperbolic functions at $x = 0$.)

b. Show that $\cosh(ix) = \cos(x)$ and that $-i\sinh(ix) = \sin(x)$.*

In Problems 30-32, use Taylor series to simplify the evaluation of the given limits. Confirm your answers by using another method to evaluate the limits.

30. $\lim\limits_{x\to 0}\dfrac{\tan^{-1} x}{x}$

31. $\lim\limits_{x\to 0}\dfrac{\cos\left(x^2\right)-1}{x^4}$

32. $\lim\limits_{x\to 0}\dfrac{1-x-e^x}{\sin x}$

In Problems 33-36, express the indefinite integral as a Taylor series.

33. $\int \cos\left(x^2\right)dx$

34. $\int \dfrac{e^x-1}{x}dx$

35. $\int \dfrac{e^x}{x}dx$

36. $\int \sqrt{1+x^3}\,dx$; No general term is required.

In Problems 37-39, express the value of the definite integral as an infinite series. Then use a partial sum (at least three terms) to approximate the value of the integral. Estimate the error in your approximation.

37. $\int_0^1 \cos\left(x^2\right)dx$

38. $\int_0^2 \dfrac{dx}{e^x}$

39. $\int_0^1 \dfrac{\cos(3x)-1}{x}dx$

40. The "sine integral function" is defined as $\mathrm{Si}(x) = \int_0^x \dfrac{\sin t}{t}dt$. It has applications in areas of engineering and signal processing.

a. Find the Maclaurin series for Si(x).

b. Express Si(1) as an infinite series.

c. How many terms do you need to include in a computation of Si(1) in order to guarantee error less than 10^{-6}?

41. The "error function" is defined as $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$. The error function is useful in many areas of applied mathematics. For example, its close relationship with the Gaussian or normal distribution curve makes it useful in computing probabilities of events that are randomly distributed.

a. Find the Maclaurin series for $\mathrm{erf}(x)$.

b. Find the Maclaurin series for $\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)$.

c. Compute $\mathrm{erf}\left(\frac{1}{\sqrt{2}}\right)$ using the third partial sum from your answer to part (b). Estimate the error in this approximation.*

42. Identify the explicit function represented by the following series. For example, if the

* These relations give an alternate way to define the sine and cosine functions for complex inputs. In practice, this strategy is used to define the trigonometric functions in terms of the exponential function: $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$ and $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$.

* The answer to part (c) is the probability that a randomly selected measurement taken from a normally distributed set will be within one standard deviation of the mean. If you compute $\mathrm{erf}\left(\frac{2}{\sqrt{2}}\right)$ and $\mathrm{erf}\left(\frac{3}{\sqrt{2}}\right)$, you will develop the 68-95-99.7% rule that is familiar to those of you who have taken statistics. However, you'll need much higher-order partial sums to compute these with sufficient accuracy.

series were $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$, you would say, "$f(x)=e^x$."

a. $1+3x+9x^2+27x^3+81x^4+\cdots$

b. $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\cdots$

c. $\frac{-x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\frac{x^{10}}{11!}+\cdots$

d. $1-\frac{25x^2}{2!}+\frac{625x^4}{4!}-\frac{15625x^6}{6!}+\frac{390625x^8}{8!}-\cdots$

43. Identify the explicit function represented by the following series.

a. $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots$

b. $2x-\frac{8x^3}{3!}+\frac{32x^5}{5!}-\frac{128x^7}{7!}+\frac{512x^9}{9!}+\cdots$

c. $8x^2-4x^4+2x^6-x^8+\frac{x^{10}}{2}\cdots$

d. $x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{x^4}{4!}+\frac{x^5}{5!}-\cdots$

44. Identify the explicit function represented by the following series.

a. $x^3-\frac{x^5}{2!}+\frac{x^7}{4!}-\frac{x^9}{6!}+\frac{x^{11}}{8!}-\cdots$

b. $x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\frac{x^{14}}{7!}+\frac{x^{18}}{9!}-\cdots$

c. $1+x-\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}-\cdots$

d. $1+\left(1-\frac{1}{2!}\right)x^2+\left(\frac{1}{2!}+\frac{1}{4!}\right)x^4+\left(\frac{1}{3!}-\frac{1}{6!}\right)x^6+\cdots$

45. Figure 10.3 shows the graph of $f(x)=\sec(x)$ along with a partial sum of its Maclaurin series. What do you expect is the radius of convergence of this Maclaurin series? What is the interval of convergence?

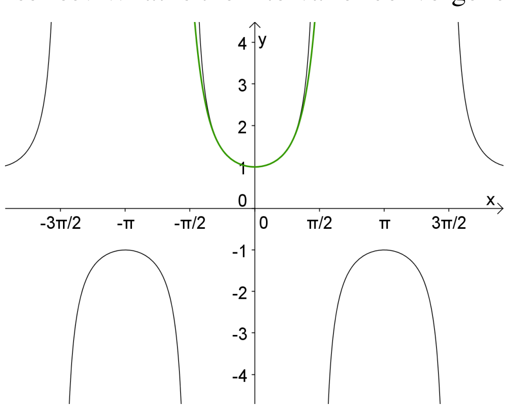

Figure 10.3

46. Figure 10.4 shows the graph of $f(x)=\dfrac{1}{1-x}$. Consider two different Taylor series for f: one centered at $x=0$ and the other centered at $x=-2$. Which has a larger radius of convergence? Explain.

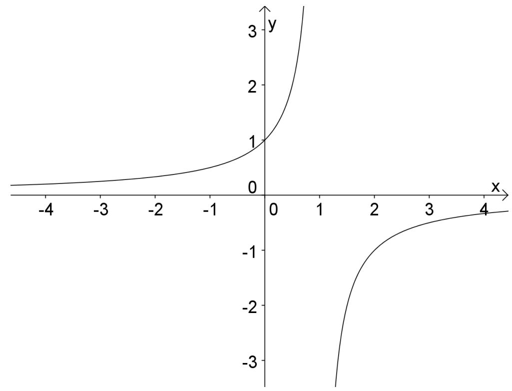

Figure 10.4

47. Figure 10.5 shows the graph of $f(x)=\dfrac{1}{x(x-1)(x-3)(x-4)}$ along with a partial sum of its Taylor series centered at $x=3.2$. As you can see, the Taylor series appears to match the function on the interval from roughly $x=3$ to $x=3.3$ or so.

a. Where in the interval (3, 4) should the Taylor series be centered so as to have the widest possible interval of convergence?

b. On the interval [-1, 5], what is the largest possible radius of convergence for a Taylor series of $f(x)$.

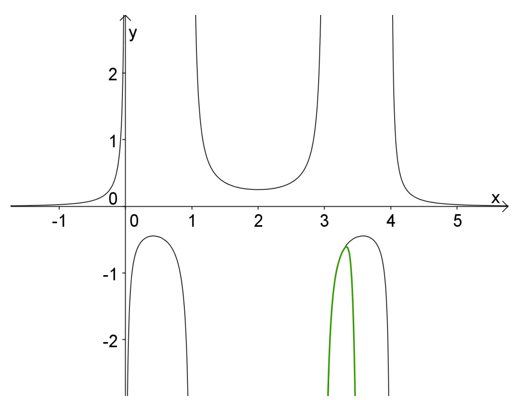

Figure 10.5

48. Use polynomial long division to obtain a Maclaurin series for the tangent function. No general term is required, but make a conjecture as to the radius of convergence of the series.

49. Show that the function

$$y = -\frac{1}{9} + \frac{x}{3} + \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2 \cdot (3x)^n}{n!}$$

solves the differential equation $y' + 3y = x$.

50. Show that the function $y = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot x^{2n-1}}{(2n)!}$ solves the differential equation $xy' + y = -\sin x$.

51. Show that the function $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n \cdot n!}$ solves the differential equation $y'' - xy' - y = 0$.

52. The **Bessel function** of order zero (there are many Bessel functions of different orders) is given by the Maclaurin series

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} \cdot (n!)^2}.$$

Show that this function satisfies the differential equation

$$x^2 y'' + xy' + x^2 y = 0.^*$$

53. Find the particular solution to the differential equation from Problem 51 (which you might prefer to write as $y'' = xy' + y$) satisfying the initial conditions $y(0) = 0$ and $y'(0) = 1$. Give your answer as a Maclaurin series. You need not include a general term.

54. Solve the initial value problem $y'' = x^3 y$, $y(0) = y'(0) = 1$. Give the first four non-zero terms. If you are feeling bolder, give the first six non-zero terms.

55. The **Hermite equation** is

$$y'' - 2xy' + \lambda y = 0$$

where λ is a constant.

a. Find a Maclaurin series solution to the Hermite equation satisfying the initial conditions $y(0) = 0$ and $y'(0) = 1$. Call your solution y_1, and give at least three non-zero terms.

b. Find a Maclaurin series solution to the Hermite equation satisfying the initial conditions $y(0) = 1$ and $y'(0) = 0$. Call your solution y_2, and give at least three non-zero terms.

c. Notice that if λ is a non-negative, even integer, then the series for either y_1 or y_2 (but not both) will terminate to give a regular polynomial. Find these polynomials when $\lambda = 4$ and $\lambda = 6$, calling the functions $h_4(x)$ and $h_6(x)$, respectively.

d. A **Hermite polynomial** of degree *n*, denoted $H_n(x)$, is determined by the equation

$$H_n(x) = k \cdot h_{2n}(x)$$

where *k* is a scaling constant chosen so that the leading coefficient of $H_n(x)$ is 2^n. Find the Hermite polynomials $H_2(x)$ and $H_3(x)$.*

e. Verify that $H_2(x)$ and $H_3(x)$ solve the Hermite eqautaion. Remember that $\lambda = 4$ for $H_2(x)$ and $\lambda = 6$ for $H_3(x)$.

* Bessel functions of various orders are important in describing waves of all kinds (from electromagnetic radiation to the vibrations of a drumhead). To generalize from order zero (in this problem) to order α, change the differential equation to $x^2 y'' + xy' + \left(x^2 - \alpha^2\right) y = 0$ and solve.

* The Hermite polynomials come up in applications of mathematics including quantum mechanics. For example, they are used in determining the wave-equation of a harmonic oscillator—basically a molecular spring. This in turn allows us to understand what frequencies of infrared light are absorbed by diatomic molecules.

56. Use the Maclaurin series for $f(x)=e^x$ to show that $2<e<4$. (If you prefer a different upper bound, that's fine.)*

57. Evaluate the following sums.

a. $3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\frac{3}{16}+\cdots$

b. $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots$

c. $1+3+\frac{9}{2!}+\frac{27}{3!}+\frac{81}{4!}+\cdots$

d. $-\frac{1}{2}-\frac{1}{4\cdot 2}-\frac{1}{8\cdot 3}-\frac{1}{16\cdot 4}-\cdots$

58. Evaluate the following sums.

a. $-5+\frac{5^3}{3!}-\frac{5^5}{5!}+\frac{5^7}{7!}-\frac{5^9}{9!}+\cdots$

b. $1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\cdots$

c. $216-36+6-1+\frac{1}{6}+\cdots$

d. $1+\ln 4+\frac{(\ln 4)^2}{2!}+\frac{(\ln 4)^3}{3!}+\cdots$

59. In Section 6, Problem 19 you proved that a series of the form $\sum_{n=1}^{\infty}\frac{n}{r^n}$ converges if $r > 1$. In Section 1, Problems 31 and 32, you found the sum of such a series. In this problem, you will take a different perspective on this series.

a. First prove that this series converges (absolutely) iff $|r|>1$.

b. Let $f(x)=\dfrac{1}{1-x}$, and let $g(x)=x\cdot f'(x)$. Give an explicit formula for $g(x)$.

c. Find the Maclaurin series for $g(x)$.

d. Show that the Maclaurin series for $g(x)$ converges when $|x|<1$.

e. Now make the substitution $x=\frac{1}{r}$ into the Maclaurin series for $g(x)$. Use this series to show that $\sum_{n=1}^{\infty}\frac{n}{r^n}$ converges for $|r|>1$. Specifically, show that $\sum_{n=1}^{\infty}\frac{n}{r^n}$ converges to $\dfrac{r}{(r-1)^2}$.

60. Let $f(x)=\dfrac{e^x-1}{x}$.

a. Find $f'(1)$ by explicitly differentiating $f(x)$ and plugging in.

b. Express $f'(x)$ as a Maclaurin series.

c. Substitute 1 for x in your answer to part (b). Use this to show that $\sum_{n=1}^{\infty}\frac{n}{(n+1)!}=1$.

61. Let $f(x)=\int_0^x te^t\,dt$.

a. Evaluate $f(1)$. (Nothing fancy yet. Just integrate.)

b. Express $f(x)$ as a Maclaurin series.

c. Substitute 1 for x in your answer to part (b). Use this to show that
$$\sum_{n=0}^{\infty}\frac{1}{(n+2)\cdot n!}=1.$$

62. Was it surprising that the series in Problems 60 and 61 had the same value? Show that they are, in fact, the same series.

63. $f(x)$ has Maclaurin series

$$1+2x+3x^2+x^3+2x^4+3x^5+x^6+\cdots=\sum_{n=0}^{\infty}c_nx^n.$$

The coefficients c_n are given by

$$c_n=\begin{cases}1, & n=3k\\ 2, & n=3k+1\\ 3, & n=3k+2\end{cases}$$

where k is an integer.

* This problem fills in a small gap of the proof that e is irrational from Problem 29 from Section 4. A slight modification of that problem may be required depending on the upper bound you select here. Also, as a hint I will point out that this problem has been done for you in an example problem somewhere in this chapter.

a. By treating this series as three inter-woven geometric series, find the sum of the series. That is, find an explicit formula for $f(x)$. (Why is rearranging the terms of the series legitimate?)

b. What is the radius of convergence of the series for *f* ? (Hint: If two power series with the same center have radii of convergence R_1 and R_2, then the radius of convergence of their sum is the smaller of R_1 and R_2. The broader theorem is that the sum of two power series converges on the intersection of the two intervals of convergence.)

c. Use this series to evaluate the sums $1+\frac{2}{2}+\frac{3}{4}+\frac{1}{8}+\frac{2}{16}+\frac{3}{32}+\cdots$ and $1-\frac{2}{3}+\frac{3}{9}-\frac{1}{27}+\frac{2}{81}-\frac{3}{243}+\cdots$.

d. Generalize the result of this problem. What are the sum and radius of convergence of the series $\sum_{n=0}^{\infty} c_n x^n$ where the coefficients c_n are given by

$$c_n = \begin{cases} 1, & n = mk \\ 2, & n = mk+1 \\ 3, & n = mk+2 \\ \vdots & \vdots \\ m, & n = mk+(m-1) \end{cases}$$

where *k* is an integer.

e. Generalize even more. Suppose the coefficients are any collection of numbers that cycle with period *m*. That is,

$$c_n = \begin{cases} c_0, & n = mk \\ c_1, & n = mk+1 \\ c_2 & n = mk+2 \\ \vdots & \vdots \\ c_{m-1}, & n = mk+(m-1). \end{cases}$$

Give the sum of this series.

f. Evaluate the following sum:

$$\frac{2}{1}+\frac{3}{5}+\frac{5}{5^2}+\frac{8}{5^3}+\frac{2}{5^4}+\frac{3}{5^5}+\frac{5}{5^6}+\frac{8}{5^7}+\cdots$$

64. Find the Taylor series for $f(x)=\frac{1}{x}$ centered at $x=1$. Also find the interval of convergence. (Hint: $\frac{1}{x}=\frac{1}{1+(x-1)}$.)

65. Find the Taylor series for $f(x)=\frac{1}{x}$ centered at $x=4$. (Hint: You can start "from scratch" by finding derivatives, or you can do some clever algebra to express $f(x)$ as the sum of a geometric series as in the hint for Problem 64.)

66. Find the Taylor series for $f(x)=\frac{1}{1-x}$ centered at the following *x*-values.

a. 2	c. 5
b. -2	d. k ($k \neq 1$)

67. Find the Taylor series for $f(x)=\frac{1}{3-x}$ centered at $x=2$.

In Problems 68-71, you will explore functions whose Taylor series converge, but not necessarily to the function they are meant to model. Sometimes this is a good thing; the Taylor series "fixes" a problems with the function. More often, though, this is an indication of something weird about the function.

68. Consider the function *f* defined as

$$f(x) = \begin{cases} \cos x, & |x| \le \pi \\ -1, & |x| > \pi. \end{cases}$$

a. Show that *f* is differentiable for all *x*.

b. Find the Maclaurin series for this function. (Hint: It shouldn't be a lot of work.)

c. Explain how you know that the series you found in part (a) converges for all *x* but does not generally converge to $f(x)$ particularly for most *x* such that $|x| > \pi$.

d. In fact, we should not expect the Maclaurin series for *f* to converge to $f(x)$ for $|x| > \pi$. Why not? (It is still unfortunate, though, that the series converges outside of the interval on which it represents its parent function.)

69. Consider the function g defined as

$$g(x)=\begin{cases} e^{-1/x^2}, & x\neq 0 \\ 0, & x=0. \end{cases}$$

a. Show that g is differentiable infinitely many times and, in particular, $g^{(n)}(0)=0$ for all n. (Hint: Induction is probably the best way to go here.)

b. Find the Maclaurin series for g and its interval of convergence.

c. Is there any non-trivial interval around the origin on which the Maclaurin series is equal to $g(x)$? (This function is a standard example of a function that infinitely differentiable (i.e., smooth), but not **analytic** (i.e., not represented by its own Maclaurin series).)

70. Let $h(x)=\dfrac{x^2-4}{x-2}$.

a. Given that this function is not defined at $x=2$, what would you expect the radius of convergence for the Maclaurin series of h to be?

b. Find the Maclaurin series for $h(x)$ and its interval of convergence. Surprised? (This behavior is typical of a function with a removable discontinuity. In fact, some texts describe a function with a removable discontinuity as one whose Taylor series "patches over" the point of discontinuity. Had there been a vertical asymptote or jump discontinuity in the graph of h, the Taylor series would have "detected" it by having a bounded interval of convergence.)

71. Let $k(x)=\dfrac{\sin x}{x}$.

a. Show that $k(x)$ has a removable discontinuity at $x=0$.

b. Formally manipulate the Maclaurin series of the sine function to obtain a Maclaurin series for $k(x)$.

c. Explain why $k(x)$ should not actually have a Maclaurin series.

d. Graph $k(x)$ and the fourth-degree partial sum of your series from part (b). Does the series seem to represent the function? Find the interval of convergence. (Again we see the Taylor series patching over a hole in the function. This time the hole is the center of the series!)

Answers to Selected Exercises

Section 1

1. a. $1, 2, 3, 4, 5$
   b. $-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}$
   c. $1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \sqrt[5]{5}$
   d. $\ln(\ln 2), \ln(\ln 3), \ln(\ln 4), \ln(\ln 5), \ln(\ln 6)$
   e. $\frac{1}{4}, \frac{3}{16}, \frac{9}{64}, \frac{27}{256}, \frac{81}{1024}$

2. a. $a_n = 2n$
   b. $a_n = \frac{(-1)^n}{n!}$
   c. $a_n = n^n$

3. 1b; converges to 0
   1c; converges to 1
   1e; converges to 0
   2b; converges to 0

5. Converges to 0 for all x

7. Diverges; harmonic

9. Diverges; fails n^{th} term test ($\lim a_n = 1$)

11. Hard to say at this point (Later we will be able to prove that this series diverges.)

13. Diverges; $\frac{\pi}{e} > 1$, so the series diverges by the geometric series test.

15. Diverges; fails n^{th} term test ($\lim a_n = 1$)

17. $\dfrac{1}{1-\frac{2}{5}} = \dfrac{5}{3}$

19. $\dfrac{1}{40}$

21. 1.5

23. $\dfrac{\frac{5}{3}}{1-\frac{1}{3}} = \dfrac{5}{2}$

25. a. $\dfrac{7}{9}$
    b. $\dfrac{37}{45}$
    c. $\dfrac{317}{999}$
    d. $\dfrac{1207}{495}$

27. Down: $\sum_{n=0}^{\infty} 6 \cdot \left(\frac{4}{5}\right)^n = 30$
    Up: $\sum_{n=1}^{\infty} 6 \cdot \left(\frac{4}{5}\right)^n = 24$
    Total: 54 feet

29. Down: $\sum_{n=0}^{\infty} 2 \cdot \left(\frac{1}{100}\right)^n = 2.020202$
    Up: $\sum_{n=1}^{\infty} 2 \cdot \left(\frac{1}{100}\right)^n = 0.020202$
    Total: 2.040404 feet

31. 2

33. $\dfrac{r}{(1-r)^2}$

35. 1.5

37. 3

39. Diverges

43. True

45. False

47. True

49. a. $\sum_{n=0}^{\infty} T \cdot \left(\frac{1}{4}\right)^n$
    b. $\dfrac{4T}{3}$

Section 2

1. $P_5(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$

3. $P_5(x) = 1 - x + x^2 - x^3 + x^4 - x^5$

5. a. $\dfrac{1}{1+x^2} \approx 1 - x^2 + x^4 - x^6$
   b. $\arctan(x) \approx x - \frac{1}{3}x^3 + \frac{1}{5}x^5$
   d. $\arctan(0.2) \approx 0.1974$ (very close)
   $\arctan(-0.6) \approx -0.5436$ (pretty close)
   $\arctan(3) \approx 42.6$ (absolutely terrible)

7. a. $P_3(x) = \frac{5}{2} + \frac{5}{4}x + \frac{5}{8}x^2 + \frac{5}{16}x^3$
   b. $P_4(x) = 3 - 3x^2 + 3x^4$
   c. $P_3(x) = 2x - 2x^3$
   d. $P_3(x) = \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3$

9. a. $P_4(x) = x^2$
   b. $P_3(x) = x^3$

11. A: 7th-degree
    B: 3rd-degree

Section 3

1. a. $P_3(x) = 1 + \frac{x-1}{2} - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$
   b. $P_4(x) = e^e + e^e(x-e) + \frac{e^e(x-e)^2}{2} + \frac{e^e(x-e)^3}{6} + \frac{e^e(x-e)^4}{24}$
   c. $P_2(x) = 1 - \frac{1}{2}x^2$
   d. $P_3(x) = 8 - 2x + 3x^2 + x^3 = f(x)$

3. a. $P_2(x) = \frac{1}{2} - \frac{x-4}{16} + \frac{3(x-4)^2}{256}$
   b. $P_3(x) = 1 - 2x^2$
   c. $P_3(x) = \frac{-1}{2} + \sqrt{3}\left(x - \frac{\pi}{3}\right) + \left(x - \frac{\pi}{3}\right)^2 - \frac{2\sqrt{3}}{3}\left(x - \frac{\pi}{3}\right)^3$
   d. $P_3(x) = \frac{1}{5} - \frac{x-5}{25} + \frac{(x-5)^2}{125} - \frac{(x-5)^3}{625}$

7. a. $P_3(x) = 0$
   b. $P_3(x) = x - x^3$
   c. $P_3(x) = \frac{-2}{5} - \frac{3(x+2)}{25} - \frac{2(x+2)^2}{125} + \frac{7(x+2)^3}{625}$
   d. $P_3(x) = x + \frac{1}{3}x^3$

9. 2.1547 (calculator value: 2.1544)

11. $P_2(x) = 2 + \frac{(x+4)^2}{2}$; $P_2(-4.2) = 2.02$
    $P_3(x) = 2 + \frac{(x+4)^2}{2} + (x+4)^3$;
    $P_3(-4.2) = 2.012$
    2.012 is probably a better estimate.

13. $P_2(x) = 6 - 2x + \frac{5}{4}x^2$

15. a. 2
    b. -1
    c. Cannot be determined
    d. 72

17. $P_2(x) = 8 + 1(x+3)$

19. $a = 2$: $P_4(x) = x$
    $a = -3$: $P_4(x) = -x$
    f is not differentiable at $x = 0$.

21. $P_2(x) = 1 + kx + \frac{k(k-1)}{2}x^2$

23. True

25. False

27. a. sinh(0) = 0; cosh(0) = 1
    b. $\dfrac{d}{dx}\sinh(x) = \cosh(x)$
    $\dfrac{d}{dx}\cosh(x) = \sinh(x)$
    c. sinh: $P_6(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!}$
    cosh: $P_6(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!}$

31. a. $K_R \approx \frac{mc^2}{2}\gamma^2$
    b. $\gamma = \frac{v}{c}$, so $K_R \approx \frac{mc^2}{2}\gamma^2 = \frac{mc^2}{2} \cdot \frac{v^2}{c^2} = \frac{1}{2}mv^2$.

33. a. $g(v_S) \approx \frac{1}{343} + \frac{1}{343^2}v_S$

b. $f_{obs} \approx f_{act}(343+v_D)\left(\frac{1}{343}+\frac{1}{343}v_S\right)$

c. $f_{obs} \approx f_{act}\left(1+\frac{v_S}{343}+\frac{v_D}{343}+\frac{v_S v_D}{343^2}\right)$

Neglecting the second-order term $\frac{v_S v_D}{343^2}$ leaves us with $f_{obs} \approx f_{act}\left(1+\frac{v_S+v_D}{343}\right)$, as desired.

Section 4

1. $e \approx \frac{65}{24} = 2.708333...$
   Error $\leq 3/120$

3. $\frac{109}{15}-\frac{4}{5} \leq e^2 \leq \frac{109}{15}+\frac{4}{5}$ or
   $6.46666... \leq e^2 \leq 8.06666...$

5. a. 0.0013333…
   b. 0.0106666…

7. a. 13th
   b. 3rd

9. 6th-degree

11. $-0.33 \leq x \leq 0.33$

13. $\left|R_5(x)\right| \leq 2.17\times 10^{-5}$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$

15. $\left|R_3(x)\right| \leq 8.333\times 10^{-6}$ on $\left[-0.1, 0.1\right]$

17. a. $f(1.4) \approx P_2(1.4) = 9.44$
    b. 0.106666…

19. a. $g(1.8) \approx P_2(1.8) = -0.24$

21. a. $\left|R_5(x)\right| \leq \frac{2}{6!}(1.3-0)^6 = 0.0134$
    b. $\left|R_5(5)\right| \leq \frac{4}{6!}(5-0)^6 = 86.806$
    c. $\left|R_5(5)\right| \leq \frac{4}{6!}(5-3)^6 = 0.3556$

Section 6

1. Converges

3. Converges

5. Converges

7. Inconclusive

9. Converges

11. Converges

13. Diverges

15. Converges

17. Diverges

21. a. 2
    b. 5
    c. Converges: -1, 0, 1
       Diverges: -8
       Cannot be determined: -5, 2, 4

23. c, d

25. No

27. 2

29. 3

31. 0

33. 3

35. 1

37. 1

39. ∞

41. 3

43. a. Converges
    b. Inconclusive
    c. Diverges
    d. Converges

45. a. Diverges
    b. Converges
    c. Diverges
    d. Inconclusive

47. a. 4
    b. Converges
    c. Converges
    d. $0 \le x \le 8$

49. a. It is not of the form $\sum c_n (x-a)^n$.
    b. $\frac{12}{25-x^2}$
    c. $(-5,-1) \cup (1,5)$

Section 7

1. Converges

3. Converges

5. Diverges

7. Converges

9. Diverges

11. Converges

13. Diverges

15. Converges

17. Diverges

19. Diverges

21. Diverges

23. Converges

25. Diverges

27. Diverges

29. Converges

31. Diverges

33. Converges

35. Diverges

37. Converges

39. Converges

41. Diverges

43. Diverges

45. Diverges

47. Converges

49. Converges

51. $p > 1$

53. $p > 1$

55. Converges

Section 8

1. Converges absolutely

3. Converges absolutely

5. Diverges

7. Converges absolutely

9. Converges absolutely

11. Diverges

13. Converges absolutely

15. Converges absolutely

17. $s_{15} = -0.3491$
    Error ≤ 0.0039

19. 20 terms

21. 12 terms

23. $0.1492 \le \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \le 0.7522$

25. Absolutely convergent series; the terms go to zero faster

27. $11.956 < s < 12.089$

29. 21^{st}-degree; 7^{th}-degree; 1^{st}-degree

31. a. 0.262461
    b. $0.26234 \le \ln(1.3) \le 0.26258$
    c. $|R_5(0.3)| \le \frac{120}{6!}(0.3)^6 = 0.0001215$

33. a. $4 \cdot \arctan(1) = 4 \cdot \frac{2}{3} = \frac{8}{3} = 2.6666\ldots$
    b. 4/5
    c. 400

39. False

41. False

43. False

45. $R = 1; -2 \le x \le 0$

47. $R = \dfrac{1}{3}; -\dfrac{1}{3} \le x < \dfrac{1}{3}$

49. $R = 3; -3 < x < 3$

51. $R = 3; -3 \le x \le 3$

53. $R = \infty; -\infty < x < \infty$

55. $R = \infty; -\infty < x < \infty$

57. $R = 1; -4 \le x < -2$

59. $R = 1; -1 \le x \le 1$

65. a. $-1 < x < 1$
    b. $f'(x) = \sum_{n=1}^{\infty} n \cdot x^{n-1}$; $-1 < x < 1$
    c. $\sum_{n=1}^{\infty} \frac{x^n}{n}$; $-1 \le x < 1$

67. a. $-4 < x < 2$
    b. $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^3}{3^n} \cdot (x+1)^{n-1}$; $-4 < x < 2$
    c. $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2}{3^n} \cdot \frac{(x+1)^{n+1}}{n+1}$; $-4 < x < 2$

69. a. $R = \infty$
    b. $f(4) \approx 0.60416666\ldots$
    c. $\dfrac{1}{384}$
    d. 8

Section 10

1. $1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$
   $-\infty < x < \infty$

3. $1 + x + x^2 + \cdots + x^n + \cdots$
   $-1 < x < 1$

5. $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$
   $-1 \le x \le 1$

7. $\frac{-1}{2} - \frac{\sqrt{3}}{2}\left(x + \frac{4\pi}{3}\right) + \frac{1}{4}\left(x + \frac{4\pi}{3}\right)^2 + \cdots$

9. $(x-2) - \frac{(x-2)^3}{3!} + \frac{(x-2)^5}{5!} - \cdots + (-1)^n \cdot \frac{(x-2)^{2n+1}}{(2n+1)!} + \cdots$

11. $x + x^2 + \frac{x^3}{2!} + \cdots + \frac{x^{n+1}}{n!} + \cdots$

13. $1 - x^2 + \frac{1}{3}x^4 - \cdots + \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!} + \cdots$

15. $2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20736}(x-8)^3 - \cdots$
    $R = 8$

17. c. 1
    d. $g(x) = 1 + \frac{5x}{2} + \frac{15x^2}{8} + \frac{15x^3}{48} + \cdots$
       $+ \frac{(5/2)(3/2)\cdots(5/2-n+1)}{n!} x^n + \cdots$
       $h(x) = 2 - 6x^2 + 12x^4 - 20x^6 + \cdots$
       $2 \cdot \frac{(-3)(-3-1)\cdots(-3-n+1)}{n!} \cdot x^{2n} + \cdots$
       $k(x) = 1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} + \cdots$
       $+ \frac{(-1/2)(-3/2)\cdots(-1/2-n+1)}{n!} x^{2n} + \cdots$
    e. $l(x) = \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \frac{5x^7}{112} - \cdots$

19. $3 - 2x + \frac{3}{8}x^2 - \frac{2}{27}x^3 + \cdots + \frac{(-1)^n \cdot (n+1)}{n^2 \cdot n!} x^n + \cdots$
    $-\infty < x < \infty$

21. $1 + \frac{(x-2)}{2} + \frac{(x-2)^2}{3} + \cdots + \frac{1}{n+1}(x-2)^n + \cdots$
    $1 \le x < 3$

23. 80; No

25. 1/24

27. $f(x) = 1x + 1x^2 + 2x^3 + 3x^4 + \cdots$

$f(x) = \sum_{n=1}^{\infty} F_n \cdot x^n$, where F_n is the n^{th} Fibonacci number

29. a. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

Both converge on $(-\infty, \infty)$.

31. -1/2

33. $C + x - \frac{1}{5 \cdot 2!} x^5 + \frac{1}{9 \cdot 4!} x^9 + \cdots$

$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1) \cdot (2n)!}$

35. $C + \ln|x| + x + \frac{1}{2 \cdot 2!} x^2 + \frac{1}{3 \cdot 3!} x^3 + \cdots$

$= C + \ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$

37. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1) \cdot (2n)!} \approx 0.9046$

with error $\leq 1.068 \times 10^{-4}$

39. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n}}{2n \cdot (2n)!} \approx -1.575$

with error ≤ 0.0203

41. a. $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1) \cdot n!}$

b. $\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{2^{(2n+1)/2} \cdot (2n+1) \cdot n!}$

c. $\operatorname{erf}\left(\frac{1}{\sqrt{2}}\right) \approx 0.6825$, error ≤ 0.0002309

43. a. $\ln(x+1)$

b. $\sin(2x)$

c. $\frac{16x^2}{2+x^2}$

d. $1 - e^{-x}$

45. $\pi/2$; $\frac{-\pi}{2} < x < \frac{\pi}{2}$

47. a. 3.5

b. 1

53. $y = x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \frac{1}{105}x^7 + \cdots$

55. a. $y_1 = x + \frac{2-\lambda}{6} x^3 + \frac{(6-\lambda)(2-\lambda)}{120} x^5 + \cdots$

$= x + \sum_{n=1}^{\infty} \frac{(2-\lambda)(6-\lambda)\cdots(4n-2-\lambda)}{(2n+1)!} x^{2n+1}$

b. $y_2 = 1 - \frac{\lambda}{2!} x^2 - \frac{\lambda(4-\lambda)}{4!} x^4 + \cdots$

$= x - \sum_{n=1}^{\infty} \frac{\lambda(4-\lambda)\cdots(4n-4-\lambda)}{(2n)!} x^{2n}$

c. $h_4(x) = 1 - 2x^2$

$h_6(x) = x - \frac{2}{3}x^3$

d. $H_2(x) = 4x^2 - 2$

$H_3(x) = 8x^3 - 12x$

57. a. 6

b. $\arctan(1) = \frac{\pi}{4}$

c. e^3

d. $\ln\left(\frac{1}{2}\right)$

59. b. $g(x) = \frac{x}{(1-x)^2}$

c. $g(x) = x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots$

61. a. 1

b. $\frac{x^2}{2 \cdot 0!} + \frac{x^3}{3 \cdot 1!} + \frac{x^4}{4 \cdot 2!} + \cdots + \frac{x^{n+2}}{(n+2) \cdot n!} + \cdots$

63. a. $f(x) = \frac{1 + 2x + 3x^2}{1 - x^3}$

b. $R = 1$

c. $\frac{22}{7}; \frac{9}{14}$

d. $f(x) = \frac{1 + 2x + 3x^2 + \cdots + mx^{m-1}}{1 - x^m}$

$R = 1$

e. $f(x) = \dfrac{c_0 + c_1 x + c_2 x^2 + \cdots + c_{m-1} x^{m-1}}{1 - x^m}$

f. $\dfrac{895}{312} = 2.8685897\ldots$

65. $\dfrac{1}{4} - \dfrac{x-4}{16} + \dfrac{(x-4)^2}{64} - \cdots + \frac{1}{4} \cdot \left(\frac{-(x-4)}{4}\right)^n + \cdots$

67. $1 + (x-2) + (x-2)^2 + \cdots + (x-2)^n + \cdots$

71. b. $1 - \frac{1}{3!} x^2 + \frac{1}{5!} x^4 - \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n} + \cdots$