# Section 1

1. a. $\frac{1!}{0!}, \frac{2!}{1!}, \frac{3!}{2!}, \frac{4!}{3!}, \frac{5!}{4!}$, which simplifies to 1, 2, 3, 4, 5
   b. Because of the $n$ in the denominator, we must start with $n = 1$.
   $\frac{\cos(\pi)}{1}, \frac{\cos(2\pi)}{2}, \frac{\cos(3\pi)}{3}, \frac{\cos(4\pi)}{4}, \frac{\cos(5\pi)}{5}$, which simplifies to $-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}$
   c. $\sqrt[0]{0}$ is undefined, so we start with $n = 1$.
   $\sqrt[1]{1}, \sqrt[2]{2}, \sqrt[3]{3}, \sqrt[4]{4}, \sqrt[5]{5}$
   d. Since ln(0) is undefined, we must start with an $n$-value that makes the inner logarithm non-zero. The smallest $n$-value that does the job is $n = 2$.
   ln(ln 2), ln(ln 3), ln(ln 4), ln(ln 5), ln(ln 6)
   e. $\frac{3^0}{4^1}, \frac{3^1}{4^2}, \frac{3^2}{4^3}, \frac{3^3}{4^4}, \frac{3^4}{4^5}$, which simplifies to $\frac{1}{4}, \frac{3}{16}, \frac{9}{64}, \frac{27}{256}, \frac{81}{1024}$
2. Answers may vary, depending on indexing. Here are possibilities.
   a. $a_n = 2n$, $n$ starting at 1
   b. $a_n = \frac{(-1)^{n+1}}{n!}$, $n$ starting at 0
   c. $a_n = n^n$, $n$ starting at 1
3. The sequence in 1b converges to 0.
   The sequence in 1c converges to 1.
   The sequence in 1e converges to 0.
   The sequence in 2b converges to 0.
4. If $|x| \geq 1$, $x^n$ will blow up as $n$ gets large. However, if $|x| < 1$, the terms diminish with increasing $n$. Therefore the sequence converges to 0 if $-1 < x < 1$.
5. This sequence converges to 0 for all $x$. In the long run, the factorial denominator will outstrip the exponential growth of the numerator, regardless of the base of the exponential.
6. a. $s_5 = 2.71\overline{6}$, $s_{10} \approx 2.71828$; the series appears to converge to $e$.
   b. $s_5 = \frac{182}{81}$, $s_{10} \approx 2.25001$; the series appears to converge to 2.25.
   c. $s_5 = 0.36\overline{6}$, $s_{10} \approx 0.367879$; this series appears to converge to $1/e$.
7. This is the harmonic series. It diverges.
8. The terms of this series grow in magnitude without bound. Therefore the series fails the $n^{\text{th}}$ term test. It diverges. Alternately, this series is geometric with $r = -2$. $|r| = 2 > 1$, so the series diverges by the geometric series test.
9. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$. This series also fails the $n^{\text{th}}$ term test. It diverges.
10. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos 0 = 1 \neq 0$. This series diverges by the $n^{\text{th}}$ term test.
11. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin 0 = 0$. This series passes the $n^{\text{th}}$ term test, but that result is inconclusive. We cannot yet say whether this series converges. (Curious? It diverges.)
12. This series is geometric with $r = \frac{e}{\pi}$. Since $e < \pi$, $|r| < 1$. This series converges by the geometric series test.
13. This time $r > 1$. The series diverges by the geometric series test.
14. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin n$ which does not exist. The series diverges by the $n^{\text{th}}$ term test.
15. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^n}{3^n + n} = 1 \neq 0$. The series diverges by the $n^{\text{th}}$ term test.
16. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n} = 0$. The $n^{\text{th}}$ term test is inconclusive. We cannot determine whether this series converges. (It diverges.)

17. $\dfrac{1}{1-\frac{2}{5}}=\dfrac{5}{3}$

18. $\dfrac{2}{1-\frac{1}{8}}=\dfrac{16}{7}$

19. $\displaystyle\sum_{n=0}^{\infty}\frac{3^n}{8^{n+2}}=\sum_{n=0}^{\infty}\frac{3^n}{8^2\cdot 8^n}=\sum_{n=0}^{\infty}\frac{1}{64}\cdot\left(\frac{3}{8}\right)^n=\frac{\frac{1}{64}}{1-\frac{3}{8}}=\frac{1}{40}$

20. $\displaystyle\sum_{n=0}^{\infty}\frac{4^{n+1}}{5^n}=\sum_{n=0}^{\infty}\frac{4\cdot 4^n}{5^n}=\sum_{n=0}^{\infty}4\cdot\left(\tfrac{4}{5}\right)^n=\frac{4}{1-\frac{4}{5}}=20$

21. $\displaystyle\sum_{n=0}^{\infty}\frac{2^{n-1}}{3^n}=\sum_{n=0}^{\infty}\frac{2^{-1}\cdot 2^n}{3^n}=\sum_{n=0}^{\infty}\tfrac{1}{2}\cdot\left(\tfrac{2}{3}\right)^n=\frac{\frac{1}{2}}{1-\frac{2}{3}}=\frac{3}{2}$

22. Since $n$ begins at 10, our initial term is $a = (3/4)^{10}$. The sum of the series is $\dfrac{\left(\frac{3}{4}\right)^{10}}{1-\frac{3}{4}}\approx 0.22525$.

23. The initial term of this series is 5/3, and $r = 1/3$. The sum of the series is $\dfrac{\frac{5}{3}}{1-\frac{1}{3}}=\dfrac{5}{2}$.

24. $\displaystyle\sum_{n=0}^{\infty}\frac{5+3^n}{4^n}=\sum_{n=0}^{\infty}\left(\frac{5}{4^n}+\frac{3^n}{4^n}\right)=\sum_{n=0}^{\infty}5\cdot\left(\tfrac{1}{4}\right)^n+\sum_{n=0}^{\infty}\left(\tfrac{3}{4}\right)^n=\frac{5}{1-\frac{1}{4}}+\frac{1}{1-\frac{3}{4}}=\frac{32}{3}$

25. a. $\displaystyle 0.77\overline{7}=0.7+0.07+0.007+\cdots=\sum_{n=0}^{\infty}0.7\cdot\left(\tfrac{1}{10}\right)^n=\frac{0.7}{1-\frac{1}{10}}=\frac{0.7}{0.9}=\frac{7}{9}$

b. $\displaystyle 0.8\overline{2}=0.8+0.02+0.002+0.0002+\cdots=0.8+\sum_{n=0}^{\infty}0.02\cdot\left(\tfrac{1}{10}\right)^n=0.8+\frac{0.02}{1-\frac{1}{10}}=\frac{4}{5}+\frac{1}{45}=\frac{37}{45}$

c. $\displaystyle 0.317\overline{317}=0.317+0.000317+0.000000317+\cdots=\sum_{n=0}^{\infty}0.317\left(\tfrac{1}{1000}\right)^n=\frac{0.317}{1-\frac{1}{1000}}=\frac{317}{999}$

d. $2.438\overline{38}=2.4+0.038+0.00038+0.0000038+\cdots$

$\displaystyle =2.4+\sum_{n=0}^{\infty}0.038\left(\tfrac{1}{100}\right)^n=2.4+\frac{0.038}{1-\frac{1}{100}}=2.4+\frac{0.038}{0.99}=\frac{12}{5}+\frac{19}{495}=\frac{1207}{495}$

26. $\displaystyle 0.\overline{9}=0.9+0.09+0.009+\cdots=\sum_{n=1}^{\infty}0.9\left(\tfrac{1}{10}\right)^n=\frac{0.9}{1-\frac{1}{10}}=\frac{0.9}{0.9}=1$. The use of Theorem 1.3 is justified here since the common ratio, 1/10, is less than 1 in absolute value.

27. There are many ways to keep track of the ups and downs of the ball. The one that my students have always preferred is to take the total amount of distance travelled downward, double it (to account for the bounces back up), and then subtract off the initial height of the ball once, since the ball travels down this initial distance but not back up.

$\displaystyle 2\cdot\sum_{n=0}^{\infty}6\cdot\left(\tfrac{4}{5}\right)^n=\frac{12}{1-\frac{4}{5}}=60$; 60 – 6 = 54 feet

28. $\displaystyle 2\cdot\sum_{n=0}^{\infty}1\cdot\left(\tfrac{1}{3}\right)^n=\frac{2}{1-\frac{1}{3}}=3$; 3 – 1 = 2 meters

29. $\displaystyle 2\cdot\sum_{n=0}^{\infty}2\cdot\left(\tfrac{1}{100}\right)^n=\frac{4}{1-\frac{1}{100}}=\frac{400}{99}\approx 4.040404$; 4.040404 – 2 = 2.040404 feet

30. $\displaystyle\sum_{n=0}^{4}1\cdot 7^n=2801$. To see why the upper limit should be 4, index the group as follows.

$n = 0$: man; $n = 1$: wives; $n = 2$: sacks; $n = 3$: cats: $n = 4$: kits

31. With the exception of the fractions whose denominators are powers of 3, every term in this series has a denominator that is divisible by 2. That suggests that each term can be obtained by multiplying some other term by 1/2. In other words, there might be some structure involving geometric series with $r = 1/2$. The terms whose denominators are not divisible by 3 at all are the most obvious; they are 1/2, 1/4, 1/8, and so forth—clearly a geometric series. If we start with the 1/3 and multiply successively by 1/2, we will generate the series 1/3 + 1/6 + 1/12, etc. This will account for every term whose denominator is divisible by 3 only once. If we want terms whose denominators have a double-factor of 3, we start with 1/9 and proceed to add on 1/18, 136, etc. In this way we can subdivide the series based on the number of times 3 divides the denominator of the term. This leads to the following rearrangement.

$$\sum_{\substack{k \text{ is divisible} \\ \text{by 2 or 3}}}^{\infty} \frac{1}{k} = \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{12}+\frac{1}{16}+\frac{1}{18}+\cdots$$

$$\begin{aligned} &= \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots \\ &\quad +\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\frac{1}{24}+\cdots \\ &\quad +\frac{1}{9}+\frac{1}{18}+\frac{1}{36}+\frac{1}{72}+\cdots \\ &\quad + \\ &\quad \vdots \end{aligned}$$

As hoped for, each line in this tableau is a geometric series with ratio 1/2. Any term of the form $\frac{1}{2^j 3^k}$ (where $j$ and $k$ are non-negative integers) can be found in this table; it will be the $j^{\text{th}}$ term in the $k^{\text{th}}$ row. ($k$ starts at 0. $j$ starts at 0 as well, except in the first line in which it starts at 1.)

We can evaluate the sum of each line individually.

First line: $\quad \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

Second line: $\quad \frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\cdots = \frac{\frac{1}{3}}{1-\frac{1}{2}} = \frac{2}{3}$

Third line: $\quad \frac{1}{9}+\frac{1}{18}+\frac{1}{36}+\cdots = \frac{\frac{1}{9}}{1-\frac{1}{2}} = \frac{2}{9}$

Had we written out the fourth line above, it would be…

Fourth line: $\quad \frac{1}{27}+\frac{1}{54}+\frac{1}{108}+\cdots = \frac{\frac{1}{27}}{1-\frac{1}{2}} = \frac{2}{27}$

We now see that the given series is equivalent to $1+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots = 1+\sum_{n=0}^{\infty}\frac{2}{3}\left(\frac{1}{3}\right)^n = 1+\frac{\frac{2}{3}}{1-\frac{1}{3}} = 1+1$.

Therefore the value of the series is 2.

32. We begin by writing out the terms and breaking them into unit fractions as suggested by the hint.

$$S = \sum_{n=1}^{\infty}\frac{n}{2^n} = \frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\cdots$$

$$= \frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\cdots$$

We now regroup, taking one term with each denominator for each grouping.

$$S = \left(\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}\cdots\right)+\left(\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\cdots\right)+\left(\tfrac{1}{8}+\tfrac{1}{16}+\cdots\right)+\left(\tfrac{1}{16}+\cdots\right)+\cdots$$

Each of these groupings is a convergent geometric series; $r = 1/2$ for all of them, and $a$ is some power of 1/2.

$$S = \left(\frac{\frac{1}{2}}{1-\frac{1}{2}}\right) + \left(\frac{\frac{1}{4}}{1-\frac{1}{2}}\right) + \left(\frac{\frac{1}{8}}{1-\frac{1}{2}}\right) + \left(\frac{\frac{1}{16}}{1-\frac{1}{2}}\right) + \cdots$$

$$= 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots$$

$$= \frac{1}{1-\frac{1}{2}} = 2$$

The value of the series is 2.

33. We proceed essentially like in Problem 32.

$$S = \sum_{n=1}^{\infty} \frac{n}{r^n} = \frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4} + \cdots$$

$$= \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^3} + \frac{1}{r^3} + \frac{1}{r^4} + \frac{1}{r^4} + \frac{1}{r^4} + \frac{1}{r^4} + \cdots$$

$$= \left(\tfrac{1}{r} + \tfrac{1}{r^2} + \tfrac{1}{r^3} + \tfrac{1}{r^4} \cdots\right) + \left(\tfrac{1}{r^2} + \tfrac{1}{r^3} + \tfrac{1}{r^4} + \cdots\right) + \left(\tfrac{1}{r^3} + \tfrac{1}{r^4} + \cdots\right) + \left(\tfrac{1}{r^4} + \cdots\right) + \cdots$$

$$= \left(\frac{\frac{1}{r}}{1-\frac{1}{r}}\right) + \left(\frac{\frac{1}{r^2}}{1-\frac{1}{r}}\right) + \left(\frac{\frac{1}{r^3}}{1-\frac{1}{r}}\right) + \left(\frac{\frac{1}{r^4}}{1-\frac{1}{r}}\right) + \cdots$$

This last step is justified because each series has ratio $1/r$, and we are given that $|r| > 1$. Therefore $|1/r|$ must be less than 1, indicating that each of the geometric series converges. Now note that $1-\frac{1}{r} = \frac{r-1}{r}$. Dividing by this quantity in each term is equivalent to multiplying by $\frac{r}{r-1}$, and we can factor this out of the summation. Continuing…

$$S = \tfrac{r}{r-1}\left(\tfrac{1}{r} + \tfrac{1}{r^2} + \tfrac{1}{r^3} + \tfrac{1}{r^4} + \cdots\right)$$

$$= \tfrac{r}{r-1} \cdot \frac{\frac{1}{r}}{1-\frac{1}{r}} = \tfrac{r}{r-1} \cdot \tfrac{r}{r-1} \cdot \tfrac{1}{r}$$

$$= \frac{r}{(r-1)^2}$$

Note that if $r = 2$, we obtain $2/1^2 = 2$, consistent with Problem 32.

34. a. $s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$s_n = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$

$s_n = 1 - \frac{1}{n+1}$

b. Sum $= \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$

35. $s_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+2}\right)$

$s_n = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right)$

$s_n = 1 + \frac{1}{2} - \frac{1}{n+2}$

$\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty}\left(1 + \frac{1}{2} - \frac{1}{n+2}\right) = \frac{3}{2}$

36. By partial fraction decomposition, $\frac{4}{n^2-1} = \frac{2}{n-1} - \frac{2}{n+1}$, so our series is $\sum\limits_{n=2}^{\infty}\left(\frac{2}{n-1} - \frac{2}{n+1}\right)$.

$$s_n = \left(\tfrac{2}{1}-\tfrac{2}{3}\right)+\left(\tfrac{2}{2}-\tfrac{2}{4}\right)+\left(\tfrac{2}{3}-\tfrac{2}{5}\right)+\left(\tfrac{2}{4}-\tfrac{2}{6}\right)+\cdots+\left(\tfrac{2}{n-1}-\tfrac{2}{n+1}\right)$$
$$s_n = \left(\tfrac{2}{1}-\cancel{\tfrac{2}{3}}\right)+\left(\tfrac{2}{2}-\cancel{\tfrac{2}{4}}\right)+\left(\cancel{\tfrac{2}{3}}-\tfrac{2}{5}\right)+\left(\cancel{\tfrac{2}{4}}-\cancel{\tfrac{2}{6}}\right)+\cdots+\left(\cancel{\tfrac{2}{n-1}}-\tfrac{2}{n+1}\right)$$
$$s_n = 2+1-\tfrac{2}{n+1}$$
$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(2+1-\tfrac{2}{n+1}\right)=3$$

37. Again, we begin by finding an expression for $s_n$.
$$s_n = \left(\tfrac{3}{\sqrt{1}}-\tfrac{3}{\sqrt{2}}\right)+\left(\tfrac{3}{\sqrt{2}}-\tfrac{3}{\sqrt{3}}\right)+\left(\tfrac{3}{\sqrt{3}}-\tfrac{3}{\sqrt{4}}\right)+\left(\tfrac{3}{\sqrt{4}}-\tfrac{3}{\sqrt{5}}\right)+\cdots+\left(\tfrac{3}{\sqrt{n}}-\tfrac{3}{\sqrt{n+1}}\right)$$
$$s_n = \left(\tfrac{3}{\sqrt{1}}-\cancel{\tfrac{3}{\sqrt{2}}}\right)+\left(\cancel{\tfrac{3}{\sqrt{2}}}-\cancel{\tfrac{3}{\sqrt{3}}}\right)+\left(\cancel{\tfrac{3}{\sqrt{3}}}-\cancel{\tfrac{3}{\sqrt{4}}}\right)+\left(\cancel{\tfrac{3}{\sqrt{4}}}-\cancel{\tfrac{3}{\sqrt{5}}}\right)+\cdots+\left(\cancel{\tfrac{3}{\sqrt{n}}}-\tfrac{3}{\sqrt{n+1}}\right)$$
$$s_n = 3-\tfrac{3}{\sqrt{n+1}}$$
$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(3-\tfrac{3}{\sqrt{n+1}}\right)=3$$

38. As in Problem 36, the trick is to use partial fractions decomposition.
$$\sum_{n=1}^{\infty}\frac{1}{n^2+n}=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\sum_{n=1}^{\infty}\left(\tfrac{1}{n}-\tfrac{1}{n+1}\right)=1$$, as we saw in Problem 34

39. While this series is telescoping, it is not convergent. To see that it is telescoping, apply a property of logarithms: $\ln\left(\tfrac{n}{n+1}\right)=\ln n-\ln(n+1)$.
$$\sum_{n=2}^{\infty}\ln\left(\tfrac{n}{n+1}\right)=\sum_{n=2}^{\infty}\left(\ln n-\ln(n+1)\right)=\left(\ln 2-\ln 3\right)+\left(\ln 3-\ln 4\right)+\left(\ln 4-\ln 5\right)+\cdots$$
The general partial sum $s_n$ is given by $s_n=\ln 2-\ln\left(n+1\right)$, but the limit as $n\to\infty$ of $s_n$ does not exist. Therefore the series diverges.

40. $s_n = \left(\arctan 1-\arctan 0\right)+\left(\arctan 2-\arctan 1\right)+\left(\arctan 3-\arctan 2\right)+\cdots+\left(\arctan(n+1)-\arctan n\right)$
$$s_n = -\arctan 0+\arctan\left(n+1\right)$$
$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(-\arctan 0+\arctan\left(n+1\right)\right)=0+\tfrac{\pi}{2}$$
The series converges to $\pi/2$.

41. Answers will vary. One example, exploiting geometric series is $a_n=2^n$, $b_n=3^n$. Then $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}2^n$ which diverges. $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}3^n$ also diverges. However, $\sum_{n=1}^{\infty}\frac{a_n}{b_n}=\sum_{n=1}^{\infty}\frac{2^n}{3^n}=\sum_{n=1}^{\infty}\left(\tfrac{2}{3}\right)^n$ is a convergent series.
42. False. The harmonic series is the classic counterexample.
43. True. This is the contrapositive of the $n^{\text{th}}$ term test.
44. False. This statement says that all series converge to zero. The *terms* of a convergent series must converge to 0, but the sum typically does not.
45. False. Let $a_n=\left(\tfrac{1}{2}\right)^n$. Then $\sum_{n=1}^{\infty}\left(\tfrac{1}{2}\right)^n$ is a convergent geometric series. However, $\sum_{n=1}^{\infty}\frac{1}{a^n}=\sum_{n=1}^{\infty}2^n$ which fails the $n^{\text{th}}$ term test.
46. False. The harmonic series is a good counterexample. If $a_n=n$, then both $\sum_{n=1}^{\infty}\frac{1}{a_n}=\sum_{n=1}^{\infty}\frac{1}{n}$ and $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}n$ diverge.

47. True. For the series $\sum_{n=1}^{\infty}\left(a_n - a_{n+1}\right)$, the $N^{\text{th}}$ partial sum is given by $s_N = \sum_{n=1}^{N}\left(a_n - a_{n+1}\right)$. We see that $s_N = \left(a_1 - a_2\right) + \left(a_2 - a_3\right) + \cdots + \left(a_N - a_{N+1}\right) = a_1 - a_{N+1}$. For the series to converge, we need $\lim_{N\to\infty} s_N$ to be finite. $\lim_{N\to\infty} s_N = \lim_{N\to\infty}\left(a_1 - a_{N+1}\right) = a_1 - \lim_{N\to\infty} a_{N+1}$. $\lim_{N\to\infty} s_N$ exists iff $\lim_{N\to\infty} a_{N+1}$ is finite. Of course, this is precisely the same as the requirement that $\lim_{n\to\infty} a_n$ is a finite constant as desired.

This is not the most general statement we can make about the convergence of telescoping series. For example, the same conclusion follows for any series that can be written in the form $\sum\left(a_n - a_{n+k}\right)$, where $k$ is fixed. We can also reorder the subtraction, which is useful in some cases.

48. a. In each stage, every segment has 1/3 of its length "erased," but then 2 new segments are drawn, each with 1/3 the length of the segment. In other words, if $l$ is the length of a segment in stage $n$, then in stage $n + 1$ that segment will be replaced with segments whose length total $\left(l - \frac{1}{3}l + 2\cdot\frac{1}{3}l\right) = \frac{4}{3}l$. In general, then, the perimeter of the snowflake grows geometrically with a common ratio of 4/3. The initial perimeter is 3. Therefore, using $p$ for perimeter, we have $p_n = 3\cdot\left(\frac{4}{3}\right)^n$.

Since $r > 1$, this sequence diverges. This tells us that the Koch snowflake has infinite perimeter.

b. The area of an equilateral triangle with side length $l$ is given by $\frac{\sqrt{3}}{4}l^2$, so the initial triangle has area $\frac{\sqrt{3}}{4}$. In stage 1, we add three triangles. Each one has 1/3 the side length of the original triangle, so they have 1/9 the area of the original triangle. (Remember that area goes like the *square* of the side length!) So the area of the stage 1 snowflake is $\frac{\sqrt{3}}{4} + 3\cdot\frac{1}{9}\cdot\frac{\sqrt{3}}{4}$.

In stage 2, we add $3\cdot 4 = 12$ new triangles. You can count this if you like (just count the triangles at one of the 6 points of the snowflake and multiply by 6). However, it should make some sense that at every stage we're adding four times as many new triangles as in the previous stage; each segment is replaced by 4 segments, and we ultimately add a triangle to each of these segments.

$$\text{Area} = \frac{\sqrt{3}}{4} + 3\cdot\frac{1}{9}\cdot\frac{\sqrt{3}}{4} + 3\cdot 4\cdot\left(\frac{1}{9}\right)^2\cdot\frac{\sqrt{3}}{4} + 3\cdot 16\cdot\left(\frac{1}{9}\right)^3\cdot\frac{\sqrt{3}}{4} + \cdots = \frac{\sqrt{3}}{4} + \sum_{n=0}^{\infty}\frac{3\sqrt{3}}{4}\cdot\frac{4^n}{9^{n+1}}$$

$$= \frac{\sqrt{3}}{4} + \sum_{n=0}^{\infty}\frac{3\sqrt{3}}{36}\cdot\frac{4^n}{9^n} = \frac{\sqrt{3}}{4} + \sum_{n=0}^{\infty}\frac{\sqrt{3}}{12}\cdot\left(\frac{4}{9}\right)^n = \frac{\sqrt{3}}{4} + \frac{\frac{\sqrt{3}}{12}}{1-\frac{4}{9}} = \frac{\sqrt{3}}{4} + \frac{9\sqrt{3}}{60} = \frac{2\sqrt{3}}{5}$$

The area of the Koch snowflake is $\frac{2\sqrt{3}}{5}$, which is finite, even though its perimeter is infinite.

49. a. Let's call Figure 1.3 "stage 0." The triangle in stage 0 has area $T$.

The new triangles in stage 1 (Figure 1.4) each have area $T/8$, and there are two of them. Therefore the additional area in stage 1 is $2\cdot\frac{T}{8} = \frac{T}{4}$.

In stage 2 (Figure 1.5), each new triangle is 1/8 the area of the triangles from stage 1, namely $\frac{1}{8}\left(\frac{T}{8}\right)$. There are four of them: 2 from each of the triangles that were new in stage 1. Therefore the new area added in stage 2 is $4\cdot\frac{T}{8^2} = \frac{T}{16}$.

As we can see, the number of new triangles doubles with every stage, so that in stage $n$ we will add $2^n$ triangles. Their areas are dwindling, though, decaying exponentially by a factor of 1/8 since the new triangles always have 1/8 the area of the triangles from the previous stage. So each new triangle in stage $n$ has area $\frac{1}{8^n}T$. Therefore stage $n$ adds a total of $2^n\cdot\frac{1}{8^n}T = \left(\frac{1}{4}\right)^n T$ units of area.

The total amount of area in the parabola, then, is $T + \frac{1}{4}T + \frac{1}{4^2}T + \cdots = \sum_{n=0}^{\infty} T\cdot\left(\frac{1}{4}\right)^n$.

b. Since the common ratio (1/4) is less than 1 in absolute value, we can easily sum the series.

Area = $\sum_{n=0}^{\infty} T\left(\frac{1}{4}\right)^n = \dfrac{T}{1-\frac{1}{4}} = \dfrac{4T}{3}$.

## Section 2

1. Answers may vary. The "right" answer is $P_5(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$.
2. For definiteness, we will start with the 5th degree Maclaurin polynomial of the sine function.

$$\sin x \approx x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5$$

$$-\cos x \approx \int\left(x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5\right)dx = \tfrac{1}{2}x^2 - \tfrac{1}{24}x^4 + \tfrac{1}{720}x^6 + C$$

$$\cos x \approx -\tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \tfrac{1}{720}x^6 + C$$

This agrees with Equation (2.2), up to the constant of integration. To fix the value of the constant, we can plug in 0 for $x$ and require equality. This gives cos(0) = 0 + $C$, or $C$ = 1. The constant of integration works out to be the constant term.

3. $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$

Therefore $P_5(x) = 1 - x + x^2 - x^3 + x^4 - x^5$.

4. a. $\dfrac{1}{1+t} \approx 1 - t + t^2 - t^3 + t^4$. Integrating both sides gives…

$$\int_0^x \frac{1}{1+t}\,dt \approx \int_0^1 \left(1 - t + t^2 - t^3 + t^4\right)dt$$

$$\ln\left|1+t\right|\Big|_0^x \approx \left(t - \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3 - \tfrac{1}{4}t^4 + \tfrac{1}{5}t^5\right)\Big|_0^x$$

$$\ln\left|1+x\right| \approx x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5$$

If we assume that $x$ is at least -1 (an assumption that will prove reasonable far down the road), then we can dispense with the absolute value bars to obtain $\ln(1+x) \approx x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5$.

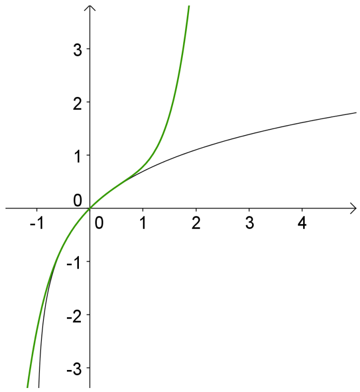
b. From Equation (2.4), $\ln(1-x) \approx -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5$. Subbing in (-$x$) for $x$…

$$\ln\left(1-(-x)\right) \approx -(-x) - \tfrac{1}{2}(-x)^2 - \tfrac{1}{3}(-x)^3 - \tfrac{1}{4}(-x)^4 - \tfrac{1}{5}(-x)^5$$

$$\ln(1+x) \approx x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5$$

c. Yes! They are the same!

d. The graphs are shown below, with the Maclaurin polynomial in green. The fit appears to be good for approximately -0.7 < $x$ < 0.7, though individual responses to this question may vary. Under no circumstances should students claim that the fit is good for $x$-values greater than 1 in magnitude.




e. ln(0.8) = ln(1 + -0.2) ≈ -0.223143

$P_5(-0.2) = (-0.2) - \frac{1}{2}(-0.2)^2 + \frac{1}{3}(-0.2)^3 - \frac{1}{4}(-0.2)^4 + \frac{1}{5}(-0.2)^5 \approx -0.223131$. Pretty close!

ln(1.8) = ln(1 + 0.8) ≈ 0.5878

$P_5(0.8) = (0.8) - \frac{1}{2}(0.8)^2 + \frac{1}{3}(0.8)^3 - \frac{1}{4}(0.8)^4 + \frac{1}{5}(0.8)^5 \approx 0.6138$. Close-ish, but not that great.

ln(5) = ln(1 + 4) ≈ 1.609

$P_5(4) = (4) - \frac{1}{2}(4)^2 + \frac{1}{3}(4)^3 - \frac{1}{4}(4)^4 + \frac{1}{5}(4)^5 \approx 158.133$. Not even remotely close.

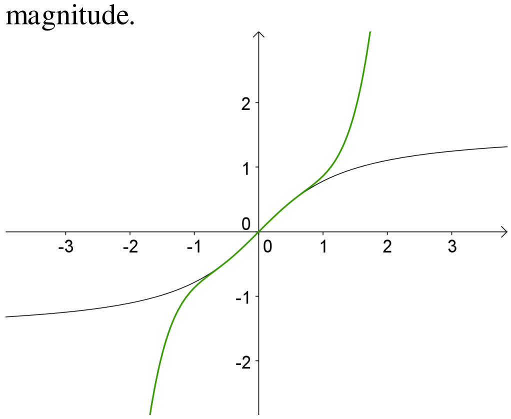
5. a. $\dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)} \approx 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 = 1 - x^2 + x^4 - x^6$

b. $\arctan x = \int_0^x \dfrac{1}{1+t^2}\,dt \approx \int_0^x (1 - t^2 + t^4 - t^6)\,dt$

$\arctan x \approx \left(t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7\right)\Big|_0^x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7$

We are asked only for the fifth-degree Macluarin polynomial. $P_5(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5$.

c. The graphs are shown below, with the Maclaurin polynomial in green. The fit appears to be good for approximately -0.75 < *x* < 0.75, though individual responses to this question may vary. Under no circumstances should students claim that the fit is good for *x*-values greater than 1 in magnitude.




d. arctan(0.2) ≈ 0.1973956

$P_5(0.2) = (0.2) - \frac{1}{3}(0.2)^3 + \frac{1}{5}(0.2)^5 \approx 0.1973973$. That's a pretty good match!

arctan(-0.6) ≈ -0.5404

$P_5(-.6) = (-.6) - \frac{1}{3}(-.6)^3 + \frac{1}{5}(-.6)^5 \approx -0.5436$. Still pretty close.

arctan(3) ≈ 1.249

$P_5(3) = (3) - \frac{1}{3}(3)^3 + \frac{1}{5}(3)^5 \approx 42.6$. That's a pretty terrible estimate.

6. a. $\sin(x) \approx x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

$\sin(2x) \approx (2x) - \frac{1}{6}(2x)^3 + \frac{1}{120}(2x)^5 = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5$

b. $2\sin x \cos x \approx 2\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right)\left(1 - \frac{1}{2}x^2\right)$

Carrying out the multiplication on a CAS and neglecting the 7th-order term, we get $2\sin x \cos x \approx 2x - \frac{4}{3}x^3 + \frac{11}{60}x^5$.

c. The approximations for $\sin(2x)$ and $2\sin(x)\cos(x)$ agree up to the third-order term. (Really up until the fourth-order term since it has a coefficient of zero in both polynomials.) This suggests that the two trigonometric expressions might be equal, which they actually are.

d. $\sin^2 x \approx \left(1 - \frac{1}{6}x^3\right)^2 = x^2 - \frac{1}{3}x^4 + \frac{1}{36}x^6$

e. $\cos^2 x \approx \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right)^2 = 1 - x^2 + \frac{1}{3}x^4 - \frac{1}{24}x^6 + \frac{1}{576}x^8$

f. $\sin^2 x + \cos^2 x \approx x^2 - \frac{1}{3}x^4 + \frac{1}{36}x^6 + 1 - x^2 + \frac{1}{3}x^4 - \frac{1}{24}x^6 + \frac{1}{576}x^8 = 1 - \frac{1}{72}x^6 + \frac{1}{576}x^8$

The eighth-degree Maclaurin polynomial for $\sin^2 x + \cos^2 x$ is essentially 1, though it includes some small non-zero contributions in the 6th and 8th degree terms. If $|x|$ is large enough, then these higher-order terms will certainly play a significant role. However, for $|x|$ sufficiently close to 0, we find that our approximation for $\sin^2 x + \cos^2 x$ is extremely close to 1, as it should be.

7. a. As a geometric series: $\dfrac{5}{2-x} = \dfrac{\frac{5}{2}}{1-\frac{x}{2}} = \dfrac{5}{2} + \dfrac{5}{2}\cdot\dfrac{x}{2} + \dfrac{5}{2}\left(\dfrac{x}{2}\right)^2 + \dfrac{5}{2}\left(\dfrac{x}{2}\right)^3 + \cdots$

$$\frac{5}{2-x} \approx \frac{5}{2} + \frac{5}{4}x + \frac{5}{8}x^2 + \frac{5}{16}x^3$$

By long division:

$$\begin{array}{r l}
 & \frac{5}{2} + \frac{5}{4}x + \frac{5}{8}x^2 + \frac{5}{16}x^3 \\ \hline
2 - x \big) & 5 + 0x + 0x^2 + 0x^3 \cdots \\
 & \underline{5 - \frac{5}{2}x} \\
 & \quad \frac{5}{2}x + 0x^2 \\
 & \quad \underline{\frac{5}{2}x - \frac{5}{4}x^2} \\
 & \qquad \frac{5}{4}x^2 + 0x^3 \\
 & \qquad \underline{\frac{5}{4}x^2 - \frac{5}{8}x^3} \\
 & \qquad\qquad \frac{5}{8}x^3
\end{array}$$

Note that the quotient is the same as what we obtained from the geometric expansion.

b. $\dfrac{3}{1+x^2} = \dfrac{3}{1-\left(-x^2\right)} = 3 - 3x^2 + 3x^4 - \cdots$

$$\frac{3}{1+x^2} \approx 3 - 3x^2 + 3x^4$$

$$\boxed{3-3x^2+3x^4}$$

$$1+x^2\overline{)3+0x^2+0x^4+\cdots}$$

$$\underline{3+3x^2}$$

$$-3x^2+0x^4$$

$$\underline{-3x^2-3x^4}$$

$$3x^4$$

c. $$\frac{2x}{1+x^2}=\frac{2x}{1-\left(-x^2\right)}=2x+2x\left(-x^2\right)+\cdots$$

$$\frac{2x}{1+x^2}\approx 2x-2x^3$$

$$\boxed{2x-2x^3}$$

$$1+x^2\overline{)2x+0x^3+\cdots}$$

$$\underline{2x+2x^3}$$

$$-2x^3$$

d. $$\frac{x}{2-x}=\frac{\frac{x}{2}}{1-\frac{x}{2}}=\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\cdots$$

$$\frac{x}{2-x}\approx\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}$$

$$\boxed{\tfrac{1}{2}x+\tfrac{1}{4}x^2+\tfrac{1}{8}x^3}$$

$$2-x\overline{)x+0x^2+0x^3+\cdots}$$

$$\underline{x-\tfrac{1}{2}x^2}$$

$$\tfrac{1}{2}x^2+0x^3$$

$$\underline{\tfrac{1}{2}x^2-\tfrac{1}{4}x^3}$$

$$\tfrac{1}{4}x^3$$

8. We begin with the fourth-degree polynomial for $f(x)=\frac{1}{1-x}$ since we will lose one from the degree due to differentiation.

$$\frac{1}{1-x}\approx 1+x+x^2+x^3+x^4$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right)\approx\frac{d}{dx}\left(1+x+x^2+x^3+x^4\right)$$

$$\frac{1}{(1-x)^2}\approx 1+2x+3x^2+4x^3$$

9. a. Based on Problem 4, $\ln(1+x)\approx x-\frac{1}{2}x^2+\frac{1}{3}x^3$.

$\ln\left(1+x^2\right)\approx x^2-\frac{1}{2}\left(x^2\right)^2+\frac{1}{3}\left(x^2\right)^3$. Since we want only a second-degree polynomial, we keep only the first term. $\ln\left(1+x^2\right)\approx x^2$

b. $\sin x\approx x-\frac{1}{6}x^3$

$\sin\left(x^3\right) \approx x^3 - \frac{1}{6}\left(x^3\right)^3$. Since we want only a third-degree polynomial, we keep only the first term.

$\sin\left(x^3\right) \approx x^3$.

10. The highest-degree polynomial should provide the best fit to the graph. Therefore the sixth-degree polynomial is B, the fourth-degree polynomial is A, and the second-degree polynomial is C.
11. The higher-degree polynomial should provide the best fit to the graph. Therefore A is the seventh-degree polynomial, and B is the third-degree.

## Section 3

1. a. $P_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$

$$\begin{array}{lllll}
f(x) = \sqrt{x} & \rightarrow & f(1) = 1 & \rightarrow & 1 \\
f'(x) = \frac{1}{2}x^{-1/2} & \rightarrow & f'(1) = \frac{1}{2} & \rightarrow & \frac{1}{2} \\
f''(x) = \frac{-1}{4}x^{-3/2} & \rightarrow & f''(1) = \frac{-1}{4} & \rightarrow & \frac{-1}{8} \\
f'''(x) = \frac{3}{8}x^{-5/2} & \rightarrow & f'''(1) = \frac{3}{8} & \rightarrow & \frac{3}{8\cdot 6} = \frac{1}{16}
\end{array}$$

b. $P_4(x) = e^e + e^e(x-e) + \frac{1}{2}e^e(x-e)^2 + \frac{1}{6}e^e(x-e)^3 + \frac{1}{24}e^e(x-e)^4$

$$\begin{array}{lllll}
f(x) = e^x & \rightarrow & f(e) = e^e & \rightarrow & e^e \\
f'(x) = e^x & \rightarrow & f'(e) = e^e & \rightarrow & e^e \\
f''(x) = e^x & \rightarrow & f''(e) = e^e & \rightarrow & \frac{1}{2}e^e \\
f'''(x) = e^x & \rightarrow & f'''(e) = e^e & \rightarrow & \frac{1}{6}e^e \\
f^{(4)}(x) = e^x & \rightarrow & f^{(4)}(e) = e^e & \rightarrow & \frac{1}{24}e^e
\end{array}$$

c. $P_2(x) = 1 - \frac{1}{2}x^2$

$$\begin{array}{lllll}
f(x) = \sqrt{1-x^2} & \rightarrow & f(0) = 1 & \rightarrow & 1 \\
f'(x) = \dfrac{-x}{\sqrt{1-x^2}} & \rightarrow & f'(0) = 0 & \rightarrow & 0 \\
f''(x) = \dfrac{-1}{\left(1-x^2\right)^{3/2}} & \rightarrow & f''(0) = -1 & \rightarrow & \dfrac{-1}{2}
\end{array}$$

d. $P_3(x) = 8 - 2x + 3x^2 + x^3$ Hm… that looks a lot like $f(x)$.

$$\begin{array}{lllll}
f(x) = x^3 + 3x^2 - 2x + 8 & \rightarrow & f(0) = 8 & \rightarrow & 8 \\
f'(x) = 3x^2 + 6x - 2 & \rightarrow & f'(0) = -2 & \rightarrow & -2 \\
f''(x) = 6x + 6 & \rightarrow & f''(0) = 6 & \rightarrow & 3 \\
f'''(x) = 6 & \rightarrow & f'''(0) = 6 & \rightarrow & 1
\end{array}$$

2. a. $P_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$

$$\begin{array}{lllll}
f(x) = \ln(x) & \rightarrow & f(1) = 0 & \rightarrow & 0 \\
f'(x) = \frac{1}{x} & \rightarrow & f'(1) = 1 & \rightarrow & 1 \\
f''(x) = \frac{-1}{x^2} & \rightarrow & f''(1) = -1 & \rightarrow & \frac{-1}{2} \\
f'''(x) = \frac{2}{x^3} & \rightarrow & f'''(1) = 2 & \rightarrow & \frac{1}{3} \\
f^{(4)}(x) = \frac{-6}{x^4} & \rightarrow & f^{(4)}(1) = -6 & \rightarrow & \frac{-1}{4}
\end{array}$$

b. Rather than taking derivatives (which will get messy due to lots of product rule), it makes more sense to start with the Maclaurin polynomial for $e^x$ and substitute in $x^2$ for $x$.

$e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$

$e^{x^2} \approx 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \frac{1}{120}x^{10}$

But we're only asked for the fifth-order polynomial, so we ignore the last three terms.

$P_5(x) = 1 + x^2 + \frac{1}{2}x^4$

c. $P_2(x) = -1 + \frac{1}{3}(x+1) + \frac{1}{9}(x+1)^2$

$$\begin{array}{ccccc} f(x) = x^{1/3} & \rightarrow & f(-1) = -1 & \rightarrow & -1 \\ f'(x) = \frac{1}{3}x^{-2/3} & \rightarrow & f'(-1) = \frac{1}{3} & \rightarrow & \frac{1}{3} \\ f''(x) = \frac{-2}{9}x^{-5/3} & \rightarrow & f''(-1) = \frac{2}{9} & \rightarrow & \frac{1}{9} \end{array}$$

d. $P_2(x) = 4 + 9(x-1) + (x-1)^2 = x^2 + 7x - 4$, if you expand it. The 2nd expression looks familiar.

$$\begin{array}{ccccc} f(x) = x^2 + 7x - 4 & \rightarrow & f(1) = 4 & \rightarrow & 4 \\ f'(x) = 2x + 7 & \rightarrow & f'(1) = 9 & \rightarrow & 9 \\ f''(x) = 2 & \rightarrow & f''(1) = 2 & \rightarrow & 1 \end{array}$$

3. a. $P_2(x) = \frac{1}{2} - \frac{1}{16}(x-4) + \frac{3}{256}(x-4)^2$

$$\begin{array}{ccccc} f(x) = x^{-1/2} & \rightarrow & f(4) = \frac{1}{2} & \rightarrow & \frac{1}{2} \\ f'(x) = \frac{-1}{2}x^{-3/2} & \rightarrow & f'(4) = \frac{-1}{16} & \rightarrow & \frac{-1}{16} \\ f''(x) = \frac{3}{4}x^{-5/2} & \rightarrow & f''(4) = \frac{3}{128} & \rightarrow & \frac{3}{256} \end{array}$$

b. Just plug $2x$ in for $x$ in the Maclaurin polynomial for $\cos(x)$.

$\cos(x) \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

$\cos(2x) \approx 1 - \frac{1}{2}(2x)^2 + \frac{1}{24}(2x)^4$

We want the third-order polynomial, so we neglect the terms of higher degree than 2.

$P_3(x) = 1 - 2x^2$

c. This time, since the center has been moved, our best bet is to start from scratch.

$P_3(x) = \frac{-1}{2} + \sqrt{3}\left(x + \frac{\pi}{3}\right) + \left(x + \frac{\pi}{3}\right)^2 - \frac{2\sqrt{3}}{3}\left(x + \frac{\pi}{3}\right)^3$

$$\begin{array}{ccccc} f(x) = \cos(2x) & \rightarrow & f\left(\frac{-\pi}{3}\right) = \frac{-1}{2} & \rightarrow & \frac{-1}{2} \\ f'(x) = -2\sin(2x) & \rightarrow & f'\left(\frac{-\pi}{3}\right) = \sqrt{3} & \rightarrow & \sqrt{3} \\ f''(x) = -4\cos(2x) & \rightarrow & f''\left(\frac{-\pi}{3}\right) = 2 & \rightarrow & 1 \\ f'''(x) = 8\sin(2x) & \rightarrow & f'''\left(\frac{-\pi}{3}\right) = -4\sqrt{3} & \rightarrow & \frac{-2\sqrt{3}}{3} \end{array}$$

d. $P_3(x) = \frac{1}{5} - \frac{1}{25}(x-5) + \frac{1}{125}(x-5)^2 - \frac{1}{625}(x-5)^3$

$$\begin{array}{ccccc} f(x) = \frac{1}{x} & \rightarrow & f(5) = \frac{1}{5} & \rightarrow & \frac{1}{5} \\ f'(x) = \frac{-1}{x^2} & \rightarrow & f'(5) = \frac{-1}{25} & \rightarrow & \frac{-1}{25} \\ f''(x) = \frac{2}{x^3} & \rightarrow & f''(5) = \frac{2}{125} & \rightarrow & \frac{1}{125} \\ f'''(x) = \frac{-6}{x^4} & \rightarrow & f'''(5) = \frac{-6}{625} & \rightarrow & \frac{-1}{625} \end{array}$$

4. An $n^{\text{th}}$-degree polynomial is its own $n^{\text{th}}$-degree Taylor polynomial.
5. The Taylor polynomial for $\ln(x)$ is just the Maclaurin polynomial for $\ln(1 + x)$ shifted by one unit.
6. a. From Problem 1c, $\sqrt{1-x^2} \approx 1 - \frac{1}{2}x^2$. Therefore $x\sqrt{1-x^2} \approx x\left(1 - \frac{1}{2}x^2\right) = x - \frac{1}{2}x^3$. $P_3(x) = x - \frac{1}{2}x^3$.

   b. As we know, $e^x \approx 1 + x + \frac{1}{2}x^2$. Therefore $xe^x \approx x\left(1 + x + \frac{1}{2}x^2\right) = x + x^2 + \frac{1}{2}x^3$. $P_3(x) = x + x^2 + \frac{1}{2}x^3$.

c. $P_3(x) = -1 + (x-2) - (x-2)^2 + (x-2)^3$

$$\begin{aligned} f(x) &= \tfrac{1}{1-x} &&\rightarrow & f(2) &= -1 &&\rightarrow & -1 \\ f'(x) &= \tfrac{1}{(1-x)^2} &&\rightarrow & f'(2) &= 1 &&\rightarrow & 1 \\ f''(x) &= \tfrac{2}{(1-x)^3} &&\rightarrow & f''(2) &= -2 &&\rightarrow & -1 \\ f'''(x) &= \tfrac{6}{(1-x)^4} &&\rightarrow & f'''(2) &= 6 &&\rightarrow & 1 \end{aligned}$$

d. We could start from scratch, on this one, but instead we will cheat.
$\ln(4-x) = \ln(1+3-x) = \ln\left(1-(x-3)\right)$. This suggests that we can just plug in $(x-3)$ for $x$ in the polynomial for $\ln(1-x)$. This is indeed the case, and the center of the polynomial will shift automatically for us!
$P_3(x) = -(x-3) - \frac{1}{2}(x-3)^2 - \frac{1}{3}(x-3)^3$

7. a. $\sin(x) \approx x - \frac{1}{6}x^3$
$x^3\sin(x) \approx x^3\left(x - \frac{1}{6}x^3\right) = x^4 - \frac{1}{6}x^6$
There are no terms of 3rd degree or lower. So $P_3(x) = 0$. This may seem odd, but zero is in fact a good approximation for values of $x^3\sin(x)$ for $x$-values near $x = 0$.

b. $\dfrac{1}{1+x^2} \approx 1 - x^2 + x^4$
$\dfrac{x}{1+x^2} \approx x\left(1 - x^2 + x^4\right) = x - x^3 + x^5$
$P_3(x) = x - x^3$

c. $P_3(x) = \frac{-2}{5} - \frac{3}{25}(x+2) - \frac{2}{125}(x+2)^2 + \frac{7}{625}(x+2)^3$

$$\begin{aligned} f(x) &= \tfrac{x}{1+x^2} &&\rightarrow & f(-2) &= \tfrac{-2}{5} &&\rightarrow & \tfrac{-2}{5} \\ f'(x) &= \tfrac{1-x^2}{\left(1+x^2\right)^2} &&\rightarrow & f'(-2) &= \tfrac{-3}{25} &&\rightarrow & \tfrac{-3}{25} \\ f''(x) &= \tfrac{2x^3-6x}{\left(1+x^2\right)^3} &&\rightarrow & f''(-2) &= \tfrac{-4}{125} &&\rightarrow & \tfrac{-2}{125} \\ f'''(x) &= \tfrac{-6x^4+36x^2-6}{\left(1+x^2\right)^4} &&\rightarrow & f'''(-2) &= \tfrac{42}{625} &&\rightarrow & \tfrac{7}{625} \end{aligned}$$

d. $\tan x = \dfrac{\sin x}{\cos x} \approx \dfrac{x - \frac{1}{6}x^3}{1 - \frac{1}{2}x^2}$

$$\begin{array}{r|l} & x + \tfrac{1}{3}x^3 \\ \hline 1 - \tfrac{1}{2}x^2 & x - \tfrac{1}{6}x^3 \\ & x - \tfrac{1}{2}x^3 \\ \hline & \quad \tfrac{1}{3}x^3 \end{array}$$

$P_3(x) = x + \frac{1}{3}x^3$

8. a. Let $f(x) = \sqrt{x}$.
b. Let the center be $a = 9$ since that's a value near $x = 10$ at which $f$ is easy to evaluate.
c. $P_3(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + \frac{1}{3888}(x-9)^3$

$$f(x)=\sqrt{x} \quad \to \quad f(9)=3 \quad \to \quad 3$$
$$f'(x)=\tfrac{1}{2}x^{-1/2} \quad \to \quad f'(9)=\tfrac{1}{6} \quad \to \quad \tfrac{1}{6}$$
$$f''(x)=\tfrac{-1}{4}x^{-3/2} \quad \to \quad f''(9)=\tfrac{-1}{108} \quad \to \quad \tfrac{-1}{216}$$
$$f'''(x)=\tfrac{3}{8}x^{-5/2} \quad \to \quad f'''(9)=\tfrac{1}{648} \quad \to \quad \tfrac{1}{3888}$$

d. $\sqrt{10} \approx P_3(10)=3+\tfrac{1}{6}-\tfrac{1}{216}+\tfrac{1}{3888}=3.16229$

9. a. Let $f(x)=\sqrt[3]{x}$.

b. Let $a$ = 8. This is still reasonably close to 10, and it is a number at which $f$ and its derivatives will be easy to evaluate.

c. $P_3(x)=2+\tfrac{1}{12}(x-8)-\tfrac{1}{288}(x-8)^2+\tfrac{5}{20736}(x-8)^3$

$$f(x)=\sqrt[3]{x} \quad \to \quad f(8)=2 \quad \to \quad 2$$
$$f'(x)=\tfrac{1}{3}x^{-2/3} \quad \to \quad f'(8)=\tfrac{1}{12} \quad \to \quad \tfrac{1}{12}$$
$$f''(x)=\tfrac{-2}{9}x^{-5/3} \quad \to \quad f''(8)=\tfrac{-1}{144} \quad \to \quad \tfrac{-1}{288}$$
$$f'''(x)=\tfrac{10}{27}x^{-8/3} \quad \to \quad f'''(8)=\tfrac{5}{3456} \quad \to \quad \tfrac{5}{20736}$$

d. $\sqrt[3]{10} \approx P_3(10)=2+\tfrac{1}{12}\cdot 2-\tfrac{1}{288}\cdot 2^2+\tfrac{5}{20736}\cdot 2^3=2.1547$

10. $P_2(x)=3-8x+\tfrac{5}{2}x^2$; $P_2(0.3)=0.825$

$P_3(x)=3-8x+\tfrac{5}{2}x^2+\tfrac{1}{3}x^3$; $P_3(0.3)=0.834$

$P_3(0.3)$ will probably give the better approximation of $f(0.3)$ since it is a higher-degree polynomial.

11. $P_2(x)=2+0(x+4)+\tfrac{1}{2}(x+4)^2$; $P_2(-4.2)=2.02$

$P_3(x)=2+\tfrac{1}{2}(x+4)^2+(x+4)^3$; $P_3(-4.2)=2.012$

$P_3(-4.2)$ will likely give the better approximation of $f(-4.2)$ since it is a higher-degree polynomial.

12. $P_3(x)=1+\tfrac{3}{4}(x-2)+\tfrac{1}{2}(x-2)^2+\tfrac{9}{32}(x-2)^3$

$$f(2)=\tfrac{3^0}{(0+1)^2}=1 \quad \to \quad 1$$
$$f'(2)=\tfrac{3^1}{(1+1)^2}=\tfrac{3}{4} \quad \to \quad \tfrac{3}{4}$$
$$f''(2)=\tfrac{3^2}{(2+1)^2}=1 \quad \to \quad \tfrac{1}{2}$$
$$f'''(2)=\tfrac{3^3}{(3+1)^2}=\tfrac{27}{16} \quad \to \quad \tfrac{9}{32}$$

13. $P_2(x)=6-2x+\tfrac{5}{4}x^2$

$$f(0)=6 \quad \to \quad 6$$
$$f'(0)=(-1)^1\cdot\tfrac{1^2+1}{1}=-2 \quad \to \quad -2$$
$$f''(0)=(-1)^2\cdot\tfrac{2^2+1}{2}=\tfrac{5}{2} \quad \to \quad \tfrac{5}{4}$$

14. Since we only know the value of $f$ at $x$ = 3, we must use $x$ = 3 as our center. This limits us to using only information about $x$ = 3. We only have $f'(3)$, and no higher derivatives. We can only write a first-order Taylor polynomial.

15. a. Since the Taylor polynomial agrees with the function at its center, $f(-1)=P(-1)=2$.

b. $\tfrac{f'(-1)}{1!}=c_1=-1 \Rightarrow f'(-1)=-1$

c. We have no information about $f$ at $x$ = 0, so we cannot determine $f''(0)$.

d. $\tfrac{f'''(-1)}{3!}=c_3=12 \Rightarrow f'''(-1)=6\cdot 12=72$

16. a. $f(4)=P(4)=5$

b. $\frac{f''(4)}{2!}=c_2=0$ because there is no quadratic term. Therefore $f''(4)=0$.

c. $\frac{f'''(4)}{3!}=c_3=1$. Therefore $f'''(4)=1\cdot 3!=6$.

d. We have no information about $f$ at $x=0$, so we cannot determine $f''(0)$.

17. Since $f$ is infinitely differentiable and its graph has an inflection point at $x=-3$, we can infer that $f''(-3)=0$. This gives the tableau below, from which we can say that $P_2(x)=8+(x+3)$.

$f(-3)=8 \quad \to \quad 8$

$f'(-3)=1 \quad \to \quad 1$

$f''(-3)=0 \quad \to \quad 0$

18. Since $g$ has a local minimum at $x=0$, its first derivative must be zero there. (Remember that $g$ is infinitely differentiable.) This implies that the coefficient of the first-order term in the Taylor polynomial must be zero; there will be no linear term in the Taylor polynomial. This eliminates (a) and (b). Since the critical point is a minimum, the second derivative of $g$ must be positive at $x=0$. This means the quadratic coefficient in the Taylor polynomial must be positive. The answer is (c).

19. For $x>0$, $|x|=x$. For $x<0$, $|x|=-x$. $f(x)$ is two polynomial functions spliced together.

Centered at 2 (which is greater than 0): $P_4(x)=x$

Centered at -3 (which is less than 0): $P_4(x)=-x$

No "work" is required to write down these Taylor polynomials. A polynomial function is its own Taylor polynomial.

$f$ is not differentiable at $x=0$, so it has no Taylor polynomial centered there.

20. From the graph, $f(2)=0$. Near $x=0$, the graph of $f$ is increasing and concave down. Therefore $f'(2)>0$ and $f''(2)<0$. These pieces of information tell us that the Taylor polynomial will have no constant term, a positive linear coefficient, and a negative quadratic coefficient. The answer is (b).

21. $P_2(x)=1+kx+\frac{k(k-1)}{2}x^2$

$$\begin{array}{rclcl}
f(x)=(1+x)^k & \to & f(0)=1^k=1 & \to & 1\\
f'(x)=k(1+x)^{k-1} & \to & f'(0)=k\cdot 1^{k-1}=k & \to & k\\
f''(x)=k(k-1)(1+x)^{k-2} & \to & f''(0)=k(k-1)\cdot 1^{k-2}=k(k-1) & \to & \frac{k(k-1)}{2}
\end{array}$$

22. a. $f(x)=\frac{1}{(1+x)^3}=(1+x)^{-3}$; $k=-3$. $P_2(x)=1-3x+\frac{-3(-4)}{2}x^2=1-3x+6x^2$.

b. $f(x)=\sqrt[5]{(1+x)^2}=(1+x)^{2/5}$; $k=2/5$. $P_2(x)=1+\frac{2}{5}x+\frac{\frac{2}{5}\cdot\frac{-3}{5}}{2}x^2=1+\frac{2}{5}x-\frac{3}{25}x^2$.

c. $f(x)=\frac{1}{\sqrt{1-x^2}}=\left(1+\left(-x^2\right)\right)^{-1/2}$. This is more complicated because of the composition. Let us begin by finding a polynomial for $g(x)=(1+x)^{-1/2}$. Then we can just substitute in $-x^2$. In $g(x)$, $k=-1/2$.

$g:\ P_2(x)=1-\frac{1}{2}x+\frac{\frac{-1}{2}\cdot\frac{-3}{2}}{2}x^2=1-\frac{1}{2}x+\frac{3}{8}x^2$

$f:\ P_2(x)=1-\frac{1}{2}\left(-x^2\right)\cancel{+\frac{3}{8}\left(-x^2\right)^2}=1+\frac{1}{2}x^2\cancel{+\frac{3}{8}x^4}$. (The crossed-out terms are of too-high degree.)

So $P_2(x)=1+\frac{1}{2}x^2$.

d. The arcsine function is the antiderivative of the function from part (c). Therefore we can obtain its Maclaurin polynomial by integrating:

$$\arcsin x=\int_0^x\frac{1}{\sqrt{1-t^2}}\,dt\approx\int_0^x\left(1+\tfrac{1}{2}t^2\right)dt=\left(t+\tfrac{1}{6}t^3\right)\Big|_0^x=x+\tfrac{1}{6}x^3$$

If you prefer a method without definite integrals and a dummy variable…

$\int \frac{1}{\sqrt{1-x^2}}\,dx \approx \int\left(1+\frac{1}{2}x^2\right)dx = x+\frac{1}{6}x^3+C$. To determine the value of $C$, we require that this polynomial match the arcsine function at $x = 0$.

$\arcsin(0) = 0+\frac{1}{6}(0)^3+C \Rightarrow C=0$. Either way, we obtain $P_3(x) = x+\frac{1}{6}x^3$

23. True. We can use the formula $\frac{f^{(k)}(a)}{k!} = c_k$, where $c_k$ is the coefficient of the $k^{\text{th}}$-degree term in the polynomial, to find the values of $f^{(k)}(a)$ from $c_k$.
24. False. The Taylor polynomial gives no information about what is happening in a function at $x$-values other than the center.
25. False. We need to divide by $k!$.
26. True. We are permitted to substitute $(x - h)$ for $x$. This automatically moves the center of the polynomial to coincide with the graphical shift.
27. a. $\sinh(0) = 0$; $\cosh(0) = 1$

    b. $\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$

    $\frac{d}{dx}\cosh x = \frac{d}{dx}\frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x$

    c. $\sinh(x)$: $P_6(x) = x+\frac{1}{6}x^3+\frac{1}{120}x^5$ $\qquad$ $\cosh(x)$: $P_6(x) = 1+\frac{1}{2}x^2+\frac{1}{24}x^4+\frac{1}{720}x^6$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)=\sinh x$ | $\rightarrow$ | $f(0)=0$ | $\rightarrow$ | $0$ | $f(x)=\cosh x$ | $\rightarrow$ | $f(0)=1$ | $\rightarrow$ | $1$ |
| $f'(x)=\cosh x$ | $\rightarrow$ | $f'(0)=1$ | $\rightarrow$ | $1$ | $f'(x)=\sinh x$ | $\rightarrow$ | $f'(0)=0$ | $\rightarrow$ | $0$ |
| $f''(x)=\sinh x$ | $\rightarrow$ | $f''(0)=0$ | $\rightarrow$ | $0$ | $f''(x)=\cosh x$ | $\rightarrow$ | $f''(0)=1$ | $\rightarrow$ | $\frac{1}{2}$ |
| $f'''(x)=\cosh x$ | $\rightarrow$ | $f'''(0)=1$ | $\rightarrow$ | $\frac{1}{6}$ | $f'''(x)=\sinh x$ | $\rightarrow$ | $f'''(0)=0$ | $\rightarrow$ | $0$ |
| $f^{(4)}(x)=\sinh x$ | $\rightarrow$ | $f^{(4)}(0)=0$ | $\rightarrow$ | $0$ | $f^{(4)}(x)=\cosh x$ | $\rightarrow$ | $f^{(4)}(0)=1$ | $\rightarrow$ | $\frac{1}{24}$ |
| $f^{(5)}(x)=\cosh x$ | $\rightarrow$ | $f^{(5)}(0)=1$ | $\rightarrow$ | $\frac{1}{120}$ | $f^{(5)}(x)=\sinh x$ | $\rightarrow$ | $f^{(5)}(0)=0$ | $\rightarrow$ | $0$ |
| $f^{(6)}(x)=\sinh x$ | $\rightarrow$ | $f^{(6)}(0)=0$ | $\rightarrow$ | $0$ | $f^{(6)}(x)=\cosh x$ | $\rightarrow$ | $f^{(6)}(0)=1$ | $\rightarrow$ | $\frac{1}{720}$ |

    The Macluarin polynomials for the hyperbolic functions are exactly like their corresponding trigonometric functions, except that these new polynomials do not alternate.

28. $\tan^{-1}x$: $P_3(x) = x-\frac{1}{3}x^3$ (We've already seen this. Don't recreate the wheel.)

    $\tanh x = \frac{\sinh x}{\cosh x} \approx \frac{x+\frac{1}{6}x^3}{1+\frac{1}{2}x^2}$: $P_3(x) = x-\frac{1}{3}x^3$

$$\begin{array}{r|l} & x-\frac{1}{3}x^3 \\ 1+\frac{1}{2}x^2 & x+\frac{1}{6}x^3 \\ & \underline{x+\frac{1}{2}x^3} \\ & \quad -\frac{1}{3}x^3 \end{array}$$

    Since the functions have the same third-order Maclaurin polynomials (in fact, they have he same fourth-order Maclaurin polynomials), their values will be close to one another for $x$-values near zero.

29. a. Rather than solve the differential equation, let's just check that $v_1(t)$ satisfies the differential equation. Left side $= mv' = m\cdot\frac{d}{dt}(gt) = m\cdot g$ = Right side. The solution checks.

    b. It will be useful to know the derivative of the hyperbolic tangent before we begin. $\frac{d}{dx}\tanh x = \frac{d}{dx}\frac{\sinh x}{\cosh x} = \frac{\cosh^2 x-\sinh^2 x}{\cosh^2 x}$. It is not hard to show from the definitions of the hyperbolic sine and cosine functions that $\cosh^2 x-\sinh^2 x = 1$. If we define the hyperbolic secant to be the reciprocal of the hyperbolic cosine, then we have $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$, which is analogous to the regular trigonometric function.

We can now move on to the differential equation.

Left side:

$$mv' = m \cdot \tfrac{d}{dt}\left(\sqrt{\tfrac{mg}{k}} \cdot \tanh\left(\sqrt{\tfrac{gk}{m}} \cdot t\right)\right)$$
$$= m \cdot \sqrt{\tfrac{mg}{k}} \cdot \sqrt{\tfrac{gk}{m}} \cdot \operatorname{sech}^2\left(\sqrt{\tfrac{gk}{m}} \cdot t\right)$$
$$= mg \cdot \operatorname{sech}^2\left(\sqrt{\tfrac{gk}{m}} \cdot t\right)$$

Right side:

$$mg - kv^2 = mg - k\left(\sqrt{\tfrac{mg}{k}} \tanh\left(\sqrt{\tfrac{gk}{m}} \cdot t\right)\right)^2$$
$$= mg - k\left(\tfrac{mg}{k} \cdot \tanh^2\left(\sqrt{\tfrac{gk}{m}} \cdot t\right)\right)$$
$$= mg - mg \tanh^2\left(\sqrt{\tfrac{gk}{m}} \cdot t\right)$$
$$= mg\left(1 - \tanh^2\left(\sqrt{\tfrac{gk}{m}} \cdot t\right)\right)$$
$$mg \cdot \operatorname{sech}^2\left(\sqrt{\tfrac{gk}{m}} \cdot t\right)$$

The last line on the right side is another hyperbolic identity, this one similar to the Pythagorean trig identity.

In any event, left and right sides match, so this function does satisfy the differential equation. (And if you thought this problem was ugly, you should see it in terms of exponentials without the use of hyperbolic functions.)

c. We already know (from Problem 28) that $\tanh x \approx x - \frac{1}{3}x^3$. We can substitute to find a Maclaurin polynomial for $v_2(t)$.

$$P_3(t) = \sqrt{\tfrac{mg}{k}} \cdot \left[\sqrt{\tfrac{gk}{m}} \cdot t - \tfrac{1}{3}\left(\sqrt{\tfrac{gk}{m}} \cdot t\right)^3\right]$$
$$= gt - \sqrt{\tfrac{mg}{k}} \cdot \tfrac{1}{3}\left(\tfrac{gk}{m}\right)^{3/2} \cdot t^3$$

If $t$ is small, then the third-order term will be negligible. In this case, we find that $v_2(t) \approx gt = v_1(t)$. In other words, $v_1(t)$ is a good approximation for $v_2(t)$ for small $t$-values. This makes sense in context. When the object has just begun falling, it is not yet moving very quickly. Therefore there will not be much air resistance; the simpler model should give good predictions of the object's velocity.

30. As we know, $\sin\theta \approx \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5$. If $\theta$ is small, then the third- and higher-order terms will be negligible. We can safely omit them if $\theta$ is sufficiently small. For such $\theta$, $\sin\theta \approx \theta$.

31. a. The key is to replace the $\frac{1}{\sqrt{1-\gamma^2}}$ term with a polynomial. We already know from Problem 22 that $\frac{1}{\sqrt{1-\gamma^2}} \approx 1 + \frac{1}{2}\gamma^2$. Therefore $K_R \approx mc^2\left(1 + \frac{1}{2}\gamma^2 - 1\right) = mc^2 \cdot \frac{1}{2}\gamma^2$. Recall, though, that $\gamma = \frac{v}{c}$. This means we can simplify further.

$$K_R \approx mc^2 \cdot \tfrac{1}{2}\left(\tfrac{v}{c}\right)^2 = mc^2 \cdot \tfrac{1}{2} \cdot \tfrac{v^2}{c^2} = \tfrac{1}{2}mc^2 = K_C$$

b. If $v$ is much smaller than $c$, then $v/c$ (otherwise known as $\gamma$) will be close to zero. In this situation, the Maclaurin polynomial from part (a) can be used as a good approximation for kinetic energy. But the polynomial from part (a) was just the classical formula for kinetic energy! Therefore, the classical model is a good approximation for the kinetic energy of an object if it is moving slowly relative to the speed of light.

32. a. We apply the result from Problem 21, treating $d/r$ as the variable and letting $k$ = -2.

$\frac{1}{\left(1+\frac{d}{r}\right)^2} \approx 1-2\cdot\frac{d}{r}+\frac{-2\cdot-3}{2}\cdot\left(\frac{d}{r}\right)^2 = 1-2\cdot\frac{d}{r}+3\cdot\left(\frac{d}{r}\right)^2$. Now by substituting $-d/r$, we find that

$\frac{1}{\left(1-\frac{d}{r}\right)^2} \approx 1+2\cdot\frac{d}{r}+3\left(\frac{d}{r}\right)^2$.

$E \approx \frac{k}{r^2}\left[\left(1+2\cdot\frac{d}{r}+3\left(\frac{d}{r}\right)^2\right)-\left(1-2\cdot\frac{d}{r}+3\left(\frac{d}{r}\right)^2\right)\right] = \frac{k}{r^2}\left[4\cdot\frac{d}{r}\right] = \frac{4kd}{r^3}$

b. The approximation in part (a) shows that the electrical field at a distance of $r$ units along the axis of the dipole varies inversely with $r^3$. The proportionality constant is $4kd$.

33. a. $f_{obs} = f_{act}\cdot\dfrac{343+v_D}{343-v_s} = f_{act}\cdot(343+v_D)\cdot\frac{1}{343-v_S}$

Expanding $\dfrac{1}{343-v_S}$ as a geometric series, we obtain $\dfrac{1}{343-v_S} = \dfrac{1}{343}\cdot\dfrac{1}{1-\frac{v_S}{343}} \approx \dfrac{1}{343}\left(1+\frac{v_s}{343}\right)$.

b. $f_{obs} \approx f_{act}\cdot(343+v_D)\cdot\frac{1}{343}\left(1+\frac{v_S}{343}\right)$

c. The rest is just algebra, until the very end.

$$\begin{aligned} f_{act}\cdot(343+v_D)\cdot\tfrac{1}{343}\left(1+\tfrac{v_S}{343}\right) &= f_{act}\cdot\left(1+\tfrac{v_D}{343}\right)\left(1+\tfrac{v_S}{343}\right) \\ &= f_{act}\cdot\left(1+\tfrac{v_D}{343}+\tfrac{v_S}{343}+\tfrac{v_S v_D}{343}\right) \\ &= f_{act}\cdot\left(1+\tfrac{v_D+v_S}{343}+\tfrac{v_S v_D}{343}\right) \\ &\approx f_{act}\cdot\left(1+\tfrac{v_D+v_S}{343}\right) \end{aligned}$$

We have omitted the last term in the parentheses because it contains the product $v_S v_D$, which is a second-order term.

# Section 4

1. For $f(x)=e^x$, $P_4(x)=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4$. We want to approximate the value of $e$, namely $e^1$. Plugging in 1 for $x$ gives $e=f(1)\approx P_4(1)=1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}=\frac{65}{24}=2.708333...$

   $\left|R_4(1)\right| \le \frac{M}{5!}(1-0)^5$. $\frac{d^5}{dx^5}e^x=e^x$, so we need a cap on the values of $e^x$ on the interval [0, 1]. $e^x$ is increasing, so its maximum value is at $x = 1$: $e^1 = e < 3$. We use 3 for $M$. The error in our approximation is no more than $\frac{3}{5!}=\frac{1}{40}$.

2. If $f(x)=\ln(1+x)$, $\ln(1.2)=f(0.2)$. We have $P_5(x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}$, so that $\ln(1.2)=f(0.2)\approx P_5(0.2)=0.2-\frac{0.2^2}{2}+\frac{0.2^3}{3}-\frac{0.2^4}{4}+\frac{0.2^5}{5}=0.18233066...$.

   $\left|R_5(0.2)\right| \le \frac{M}{6!}\cdot(0.2-0)^6=\frac{M}{11250000}$ where $M$ is a bound for $f^{(6)}(x)$ on [0, 0.2]. $f^{(6)}(x)=\frac{-120}{(x+1)^6}$. This function is increasing, but negative, on [0, 0.2]. It will therefore take on its greatest value in magnitude at $x = 0$. (Look at a graph). $f^{(6)}(0)=-120$, so a suitable value for $M$ is 120. This means that $\left|R_5(0.5)\right| \le \frac{120}{11250000}=\frac{1}{93750}=0.00001066...$

3. We let $f(x)=e^x$, and we seek an approximation for $f(2)$. $P_5(x)=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5$, and it follows that $P_5(2)=1+2+\frac{2^2}{2}+\frac{2^3}{6}+\frac{2^4}{24}+\frac{2^5}{120}=\frac{109}{15}=7.26666...$.

$\left|R_5(2)\right| \le \frac{M}{6!}(2-0)^6 = \frac{4M}{45}$. $f^{(6)}(x) = e^x$, which is an increasing function. On the interval [0, 2], its maximum occurs at $x = 2$, suggesting $e^2$ for $M$, but that is the number we are trying to approximate. However, since $e < 3$, $e^2 < 9$. So we take 9 for $M$. $\left|R_5(2)\right| \le \frac{4 \cdot 9}{45} = \frac{4}{5}$.

Putting it all together, $\frac{109}{15} - \frac{4}{5} \le e^2 \le \frac{109}{15} + \frac{4}{5}$ or $\frac{97}{15} \le e^2 \le \frac{121}{15}$ or $6.466... \le e^2 \le 8.066...$ (Not a very tight bound, but 2 is not particularly close to 0.)

4. The approximation would be based on $P_3(x)$, so the error will be an estimate on $\left|R_3(-0.3)\right|$.
   $\left|R_3(-0.3)\right| \le \frac{M}{4!} \cdot \left|-0.3-0\right|^4 = \frac{27M}{80{,}000}$. Since the sine function and all its derivatives are bounded by 1, we can use 1 for $M$. $\left|R_3(-0.3)\right| \le \frac{27}{80{,}000} = 0.0003375$
5. $\left|R_2(x)\right| \le \frac{M}{3!}(x-0)^3$, where $M$ is a bound for $f^{(3)}(x)$ on the interval [0, 0.1] for part (a) or on the interval [-0.2, 0] for part (b). $f^{(3)}(x) = 8\sin\left(2x + \frac{\pi}{3}\right)$. The sine factor is bounded by 1, but because of the coefficient, we must use 8 for $M$. This $M$-value works on any interval.
   a. $\left|R_2(0.1)\right| \le \frac{8}{3!}(0.1-0)^3 = 0.001333...$
   b. $\left|R_2(-0.2)\right| \le \frac{8}{3!} \cdot \left|-0.2-0\right|^3 = 0.0106666...$
6. a. $\left|R_2(1)\right| \le \frac{M}{3!} \cdot \left|1 - \frac{\pi}{3}\right|^3$. We can continue to use 1 for $M$ as sin($x$) and all its derivatives are bounded by 1 on all intervals. Therefore $\left|R_2(1)\right| \le \frac{1}{6} \cdot \left|1 - \frac{\pi}{3}\right|^3 = 0.000017523 \approx 2 \times 10^{-5}$.
   b. Now we want $\left|R_n(1)\right| \le 10^{-9}$. In general, $\left|R_n(1)\right| \le \frac{1}{(n+1)!} \cdot \left|1 - \frac{\pi}{3}\right|^{n+1}$. To guarantee the required accuracy, we set $\frac{1}{(n+1)!} \cdot \left|1 - \frac{\pi}{3}\right|^{n+1} \le 10^{-9}$ and solve by consulting a table of values. We determine that if $n = 5$, we are sure to compute sin(1) with the desired accuracy. We should use a fifth-degree polynomial. (As it turns out, a fourth-degree polynomial computes sin(1) within $10^{-9}$ of its actual value. However, we would not predict that based on the Lagrange error bound. Remember that the error bound gives an upper bound on the amount of error to expect. There may actually be much less error than what you compute using the Lagrange error bound, but we cannot count on that.)
7. For the Macluarin polynomial, $\left|R_n(3)\right| \le \frac{M}{(n+1)!} \cdot (3-0)^{n+1}$. Taking $M$ to be 1 since the sine function and all its derivatives are bounded by 1 on all intervals, this simplifies to $\left|R_n(3)\right| \le \frac{3^{n+1}}{(n+1)!}$. This expression is first less than 0.0001 when $n = 13$, indicating that we need a $13^{\text{th}}$-degree polynomial.
   If instead we center our polynomial at $x = \pi$, then $\left|R_n(3)\right| \le \frac{M}{(n+1)!} \cdot \left|3 - \pi\right|^{n+1}$. Again, we take $M = 1$ and look for when $\frac{\left|3-\pi\right|^{n+1}}{(n+1)!}$ is first less than 0.0001. This happens when $n = 3$, indicating that we need a $3^{\text{rd}}$-degree polynomial.
8. $\left|R_n(1)\right| \le \frac{M}{(n+1)!} \cdot (1-0)^{n+1} = \frac{M}{(n+1)!}$. Since the cosine function and all its derivatives are bounded by 1 on all intervals, we can take $M = 1$. This means that $\left|R_n(x)\right| \le \frac{1}{(n+1)!}$, which is about as simple an expression for the error bound as we're likely to see. We would like it to be less than 0.0001. Consulting a table of values, we see that this happens when $n = 7$, so we need a $7^{\text{th}}$-orderMaclaurin polynomial. But wait! The cosine function's Maclaurin polynomials have only even-degree terms; the $7^{\text{th}}$-order polynomial is the same as the $6^{\text{th}}$-order polynomial. Unfortunately, the $6^{\text{th}}$-degree polynomial is not guaranteed to give the desired accuracy based on the Lagrange error bound. We err on the side of caution and use an $8^{\text{th}}$-degree Maclaurin polynomial. (As it turns out, the $6^{\text{th}}$-degree polynomial *is* good enough, but there is no way to know this based on the Lagrange error bound.)

9. We want an estimate on $\left|R_n\left(\frac{3}{4}\right)\right|$. As we know, $\left|R_n\left(\frac{3}{4}\right)\right| \le \frac{M}{(n+1)!}\cdot\left|\frac{3}{4}-1\right|^{n+1}$, but we need an $M$-value. For the interval $\left[\frac{3}{4},1\right]$. Unlike the sine, cosine, and exponential functions, the bound for $M$ will depend on the derivative used. An examination of several derivatives of $f(x)=\ln(x)$ indicates the following two facts. (1) A formula for the $n^{\text{th}}$ derivative for $n \ge 1$ is $f^{(n)}(x)=(-1)^{n+1}\cdot\frac{(n-1)!}{x^n}$. (2) When the coefficient $(-1)^{n+1}$ is negative, $f^{(n)}$ is an increasing function, and when $(-1)^{n+1}$ is positive, $f^{(n)}$ is decreasing. In either case, the largest value of $f^{(n)}(x)$ in magnitude will occur at the left endpoint of the interval: in this case at $x=\frac{3}{4}$. A bound for the $(n+1)^{\text{st}}$ derivative, then, is $M=\frac{((n+1)-1)!}{(3/4)^{n+1}}=n!\cdot\left(\frac{4}{3}\right)^{n+1}$. Coming back to the remainder term, we have $\left|R_n\left(\frac{3}{4}\right)\right| \le n!\cdot\left(\frac{4}{3}\right)^{n+1}\cdot\frac{1}{(n+1)!}\cdot\left(\frac{1}{4}\right)^{n+1}=\frac{1}{n\cdot 3^{n+1}}$. Consulting a table of values for this last expression, we find that it is first less than 0.0001 when $n$ = 6. Therefore we need a 6th-degree Taylor polynomial.
10. Using the result of Section 3, Problem #21, $\sqrt{1+x}=(1+x)^{1/2}\approx 1+\frac{1}{2}x+\frac{(1/2)(-1/2)}{2!}x^2$. Therefore $P_2(x)=1+\frac{1}{2}x-\frac{1}{8}x^2$. If $f(x)=\sqrt{1+x}$, then $\sqrt{1.4}=\sqrt{1+0.4}=f(0.4)\approx P_2(0.4)=1.18$. To put bounds on this approximation, we compute the Lagrange error bound: $\left|R_2(0.4)\right|$. This will require finding a bound for $f^{(3)}(x)=\frac{3}{8(1+x)^{5/2}}$ on the interval [0, 0.4]. $f^{(3)}$ is a decreasing function, so its max occurs at the left end of the interval. $f^{(3)}(0)=\frac{3}{8}$, so we use that for $M$. $\left|R_2(0.4)\right| \le \frac{3}{8\cdot 3!}\cdot(0.4-0)^3=0.004$. We conclude that $1.18-0.004\le\sqrt{1.4}\le 1.18+0.004$ or $1.176\le\sqrt{1.4}\le 1.184$.
11. $\left|R_3(x)\right| \le \frac{M}{4!}\left|x-0\right|^4$. Since the sine function and all its derivatives are bounded by 1, we let $M$ = 1, giving $\left|R_3(x)\right| \le \frac{|x|^4}{4!}$, and we would like this to be less than 0.0005. $\frac{|x|^4}{4!}<0.0005 \Rightarrow |x|^4<0.012$ $\Rightarrow |x|<0.33$. We will have the required accuracy for $x$ such that $-0.33 < x < 0.33$.
12. $\left|R_4(x)\right| \le \frac{M}{5!}\left|x-0\right|^5$. Since the cosine function and all its derivatives are bounded by 1, we let $M$ = 1, giving $\left|R_4(x)\right| \le \frac{|x|^5}{5!}$. We would like this to be less than 0.00005 in absolute value to ensure accuracy to four decimal places. $\frac{|x|^5}{5!}<0.00005 \Rightarrow |x|^5<0.006 \Rightarrow |x|<0.359$. The required accuracy is guaranteed for $-0.359 < x < 0.359$.
13. The error, $\left|R_5(x)\right|$, will be bounded by $\frac{M}{6!}\cdot\left|x-0\right|^6$. Since the sine function and all its derivatives are bounded by 1, we let $M$ = 1. Our $x$-values range from $\frac{-1}{2}$ to $\frac{1}{2}$. The maximum value of $\left|x-0\right|^6$ for these $x$-values is $(1/2)^6$. Therefore an (over)estimate of the error from a fifth-degree Maclaurin polynomial on this interval is $\frac{1}{6!}\cdot\left(\frac{1}{2}\right)^6=2.17\times10^{-5}$.
14. $\left|R_2(x)\right|$ will be bounded by $\frac{M}{3!}\cdot\left|x-0\right|^3$. We need to pick values for $M$ and $x$ to plug into this expression. Our $x$-values range from -0.2 to 0.2. To make sure that our error bound overestimates the actual error in the computation, we choose an $x$-value that will make the factor $\left|x-0\right|^3$ largest. Because of the absolute value bars, either +0.2 or -0.2 will do for this. $M$ will be a bound on the values of $f^{(3)}(x)=\frac{2}{(1+x)^3}$ as $x$ ranges from -0.2 to +0.2. The largest value occurs at $x$ = -0.2 and is $f^{(3)}(-0.2)=\frac{2}{0.8^3}=3.90625$. Therefore $\left|R_2(x)\right| \le \frac{3.90625}{3!}\cdot(0.2)^3=0.005208$ for $x$ in the interval $\left[-0.2,0.2\right]$. That is an (over)estimate on the greatest possible error we expect to see from the 2nd-degree Maclaurin polynomial.

Note that this was a worst-case scenario analysis. We want to overestimate the actual error that we will observe. (Underestimating expected error is never a good idea.) To do this, we looked at each piece of the Lagrange error bound formula individually and plugged in values that made that piece as big as we would reasonably expect it to be. We've been a little sloppy about the intervals as a consequence. Normally for Lagrange error bound use, we're supposed to look at an interval in which the center of the polynomial is one of the endpoints. In this case, because we're looking at $x$-values on *both* sides of the center, we relaxed that to consider all possible $x$-values at once. The result is that our error estimate will probably be hugely exaggerated for some $x$-values in the interval. But that's okay; we're looking for the largest possible error we expect to see, and I do not believe we will see error greater than 0.0052.

15. $|R_3(x)|$ will be bounded by $\frac{M}{4!}\cdot|x-0|^4$. $M$ is a bound on $f^{(4)}(x)=e^x$ on the interval [-0.1, 0.1]. The greatest value taken on by $f^{(4)}(x)=e^x$ will be at the right endpoint of the interval since $e^x$ is an increasing function. Therefore, we would use $M=e^{0.1}$, but who knows what this value is. We could take the easy way out and let $M = 3$ ($e < 3$, so $e^{0.1} < 3$ as well), but let's do a little better. $e^{0.1}<e^{0.5}<4^{0.5}=\sqrt{4}=2$. Let's take $M = 2$. The largest value of $|x-0|^4$ for the interval in question is $0.1^4$. Putting it all together, we have $|R_3(x)|\le\frac{2}{4!}\cdot 0.1^4=8.333\times10^{-6}$. This is the largest error we expect to see. (See the note in the solution to Problem 14 about overestimating error.)

16. $\sqrt{10}$ : We used a 3rd-degree Taylor polynomial centered at $x = 9$ to approximate $f(x)=\sqrt{x}$. Therefore $|R_3(10)|\le\frac{M}{4!}(10-9)^4=\frac{M}{24}$. To find a value for $M$, we need the biggest values (in magnitude) taken on by $f^{(4)}(x)=\frac{-15}{16x^{7/2}}$ on [9, 10]. We want to make the denominator small to make the fraction big, so we pick $x = 9$. $|f^{(4)}(9)|=0.0004287$. Therefore $|R_3(10)|\le\frac{0.0004287}{24}\approx1.8\times10^{-5}$.

    $\sqrt[3]{10}$ : We used a 3rd-degree Taylor polynomial centered at $x = 8$ to approximate $f(x)=\sqrt[3]{x}$. Therefore $|R_3(10)|\le\frac{M}{4!}(10-8)^4=\frac{2M}{3}$. For this function, $f^{(4)}(x)=\frac{-80}{81x^{11/3}}$. To find $M$, we will again plug in the smallest $x$-value we can, namely 8. $|f^{(4)}(8)|=0.000482$. Therefore $|R_3(10)|\le3.2\times10^{-4}$.

17. a. $P_2(x)=8+4(x-1)-\frac{2}{2}(x-1)^2$. $f(1.4)\approx P_2(1.4)=9.44$.
    b. $|R_3(1.4)|\le\frac{10}{4!}(1.4-1)^4=0.1066...$

18. a. $P_2(x)=2-3x+\frac{4}{2}x^2$. $f(-1)\approx P(-1)=7$
    b. $|R_2(-1)|\le\frac{2}{3!}\cdot|-1-0|^3=\frac{1}{3}$. Therefore, the actual value of $f(-1)$ is between 7 – 1/3 and 7 + 1/3. Hence, the maximum possible value of $f(-1)$ is $7\frac{1}{3}$, which is less than 8.75. $\therefore f(-1)\ne 8.75$.

19. a. $P_2(x)=0+2(x-2)+\frac{8}{2}(x-2)^2$. $g(1.8)\approx P_2(1.8)=-0.24$.
    b. $|R_2(x)|\le\frac{5}{3!}|1.8-2|^3=0.0066...$. The maximum possible value of $g(1.8)$ is -0.24 + 0.0066… = $-0.2333...<0$. We conclude that $g(1.8)<0$.

20. $P_1(x)=2+5(x+3)$. $h(-2.5)\approx P_1(-2.5)=2+5(-2.5+3)=4.5$. $|R_1(-2.5)|\le\frac{1}{2!}(-2.5+3)^2=0.125$. Therefore $4.5-0.125\le h(-2.5)\le4.5+0.125$ or $4.375\le h(-2.5)\le4.625$.

21. a. On the interval [0, 1.3], the maximum value of $f^{(6)}(x)$ is 2. Therefore, we take 2 for $M$. $|R_5(x)|\le\frac{2}{6!}(1.3-0)^6=0.0134$. This is the maximum possible error in using the 5th-degree Maclaurin polynomial.
    b. On the interval [0, 5], the maximum *absolute* value of $f^{(6)}(x)$ is 4; this is our $M$-value. $|R_5(5)|\le\frac{4}{6!}(5-0)^6=86.806$. Ugh. That's a lot of error.

c. The maximum absolute value of $f^{(6)}(x)$ on [3, 5] is still 4. The only change from part (b) is a much-needed adjustment to the center of the polynomial. $|R_5(5)| \le \frac{4}{6!}(5-3)^6 = 0.3556$.

22. a. The division below shows several iterations of the long division algorithm. The boxed terms are the remainders from one iteration; they are what the remainder would be if the division were stopped at that stage. (Note: In this paragraph, "remainder" is being used in the sense of division, not in the sense of Lagrange remainder.)

$$\begin{array}{r l}
 & 1-x^2+x^4-x^6 \\
1+x^2 \,\big) & 1+0x^2+0x^4+0x^6\cdots \\
 & \underline{1+x^2} \\
 & \quad \boxed{-x^2}+0x^4 \\
 & \quad \underline{-x^2-x^4} \\
 & \qquad\quad \boxed{x^4}+0x^6 \\
 & \qquad\quad \underline{x^4+x^6} \\
 & \qquad\qquad \boxed{-x^6}
\end{array}$$

Based on this division, several possible representations of $f(x)$ are…

$1-\frac{x^2}{1+x^2}$

$1-x^2+\frac{x^4}{1+x^2}$

$1-x^2+x^4-\frac{x^6}{1+x^2}$

As we can see, the left-over term after the polynomial has terminated – the remainder in the sense of Lagrange remainder – is given by $R_{2n}(x)=(-1)^{n+1}\cdot\frac{x^{2n+2}}{1+x^2}$. In other words, $|R_{2n}(x)|=\frac{x^{2n+2}}{1+x^2}$.

b. We know that $\frac{1}{1+x^2}$ can be expanded as a geometric series as $1-x^2+x^4-x^6+\cdots$. If we stop this polynomial at some point, part (a) indicates what the remainder will be. Stopping the polynomial at degree $2n$ and incorporating the remainder from part (a), we have $\frac{1}{1+x^2}=1-x^2+x^4+\cdots+(-1)^n x^{2n}+(-1)^{n+1}\cdot\frac{x^{2n+2}}{1+x^2}$ as desired.

23. a. $\arctan x \approx x-\frac{1}{3}x^3+\frac{1}{5}x^5-\cdots+\frac{(-1)^n}{2n+1}\cdot x^{2n+1}$.

Now plug in 1 for $x$: $\arctan(1)=\frac{\pi}{4}\approx 1-\frac{1}{3}+\frac{1}{5}-\cdots+\frac{(-1)^n}{2n+1}$.

Multiply through by 4, and we are finished: $\pi \approx 4\cdot\left(1-\frac{1}{3}+\frac{1}{5}-\cdots+\frac{(-1)^n}{2n+1}\right)$.

b. Using 5 terms, we have $\pi \approx 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}\right)=3.3397$. This is not actually a very good estimate.

c. Problem 23b told us that $\frac{1}{1+x^2}=1-x^2+x^4-\cdots+(-1)^n x^{2n}+(-1)^{n+1}\cdot\frac{x^{2n+2}}{1+x^2}$. Integrating both sides, we find $\arctan x = x-\frac{1}{3}x^3+\frac{1}{5}x^5-\cdots+\frac{(-1)^n}{2n+1}\cdot x^{2n+1}+(-1)^{n+1}\cdot\int_0^x \frac{t^{2n+2}}{1+t^2}dt$. ($t$ is just a dummy variable.)

All but the last term of the right side of this last equation make up the $(2n+1)^{\text{st}}$-degree Maclaurin polynomial for the arctangent function. The last term is the remainder. Taking its absolute value, we have the desired result: $|R_{2n+1}(x)|=\int_0^x \frac{t^{2n+2}}{1+t^2}dt$.

d. For all $t$, $1+t^2 \geq 1$. Therefore $\frac{t^{2n+2}}{1+t^2} \leq t^{2n+2}$; dividing $t^{2n+2}$ by a number at least 1 will make it smaller. (Note that $t^{2n+2}$ is positive for all integers $n$.) Now $t^{2n+2} \leq \frac{t^{2n+2}}{1+t^2} \Rightarrow \int_0^x t^{2n+2}dt \leq \int_0^x \frac{t^{2n+2}}{1+t^2}dt$ as long as $x > 0$; this is a property of definite integrals. Since $\left|R_{2n+1}(x)\right| = \int_0^x t^{2n+2}dt$, it follows that $\left|R_{2n+1}(x)\right| \leq \int_0^x t^{2n+2}dt$.

e. $\int_0^x t^{2n+2}dt = \frac{1}{2n+3}t^{2n+3}\Big|_0^x = \frac{1}{2n+3}\cdot x^{2n+3}$. Therefore $\left|R_{2n+1}(x)\right| \leq \frac{1}{2n+3}\cdot x^{2n+3}$.

f. We are interested in $\left|R_{2n+1}(1)\right|$ which is $\frac{1}{2n+3}$, but be careful of the 4 in Equation (1)!!! If we just set $\left|R_{2n+1}(1)\right|$ less than 0.01 and solve, we will find an $n$-value that will approximate $\pi/4$ with error less than 0.01. But when we multiply by 4, the error might be as high as 0.04. To account for this, we need the remainder to be less than 0.01/4 = 0.0025.
$\frac{1}{2n+3} < 0.0025 \Rightarrow \frac{1}{0.0025} < 2n+3 \Rightarrow 400 < 2n+3 \Rightarrow 397 < 2n \Rightarrow n \geq 199$.
We need to let $n$ = 199 in order to estimate $\pi$ with the desired accuracy. That's a lot of terms for not very much accuracy. There are far better ways to approximate $\pi$.

24. We take the case of $x > 0$ first. The error in computing $e^x$ with a Maclaurin polynomial of degree $n$ will be $\left|R_n(x)\right| \leq \frac{M}{(n+1)!}\cdot x^{n+1}$, where $M$ is an upper bound for the $(n + 1)^{\text{st}}$ derivative of $e^x$ on the interval $[0, x]$. (Absolute value bars on $x$ are unnecessary since $x > 0$.) The $(n + 1)^{\text{st}}$ derivative of $e^x$ is just $e^x$, though. Therefore we are looking for a bound on the value of $e^x$ to use for $M$. Since $e^x$ is an increasing function, it obtains its maximum at the right hand endpoint of the interval in question. In other words, $e^t \leq e^x$ for all $t$ in the interval $[0, x]$. Since $e < 3$, it follows that $e^x < 3^x$ for any positive $x$-value. The upshot of all this is that we can use $3^x$ for $M$. Then we have $\left|R_n(x)\right| \leq \frac{3^x \cdot x^{n+1}}{(n+1)!}$, as desired.
If $x$ is negative, $e^x < e^0 = 1$, again because $e^x$ is an increasing function. This means that we can use 1 for $M$. Now $\left|R_n(x)\right| \leq \frac{1}{(n+1)!}\left|x-0\right|^{n+1} = \frac{|x|^{n+1}}{(n+1)!}$, as desired.
Factorials grow larger than exponentials in the long run. Therefore, if we take $n$ to be large enough, the error in computing $e^x$ will be small; in fact, it can be made smaller than any desired tolerance.

25. The Lagrange remainder for using an $n^{\text{th}}$-degree Maclaurin approximation for $e^x$ is $R_n(x) = \frac{1}{(n+1)!}\cdot f^{(n+1)}(z)\cdot(x-0)^{n+1}$. But $f^{(n+1)}(x) = e^x$ for all $n$, and $e^x$ is positive for all $x$. This means that every factor in the remainder term is positive when $x$ is positive; $R_n(x)$ is positive for all $n$ and all $x > 0$. Since $f(x) = P_n(x) + R_n(x)$, and $R_n(x)$ is positive, $P_n(x)$ must be too small; it underestimates the value of $e^x$ when $x > 0$.

26. a. $\left|R_n(20)\right| \leq \frac{M}{(n+1)!}\cdot 20^{n+1}$. Since $f(x) = \cos x$ and all its derivatives are bounded by 1, we can use 1 for $M$. Therefore $\left|R_n(20)\right| \leq \frac{20^{n+1}}{(n+1)!}$. We require that this expression be less than $10^{-3}$. Using a table to solve $\frac{20^{n+1}}{(n+1)!} \leq 10^{-3}$, we see that we require $n$ to be at least 58. We need a $58^{\text{th}}$-degree Maclaurin polynomial for the required accuracy.
An $n^{\text{th}}$-degree Maclaurin polynomial for the cosine function is missing all odd-degree terms. In particular, a $58^{\text{th}}$-degree polynomial has $(58 \div 2) + 1 = 30$ terms.

b. We can simply subtract off multiples of $2\pi$ from 20 until we get a number in the desired range. $t = 20 - 6\pi \approx 1.1504$ does the trick.

$\left|R_n(t)\right| \le \frac{1}{(n+1)!} \cdot t^{n+1}$. Using a table to solve $\frac{1}{(n+1)!} \cdot t^{n+1} \le 10^{-3}$, we find that we need $n$ to be at least 6, a much more manageable number!

c. Instead of attempting to approximate sin(100), we will find a smaller number $t$ such that $\sin(t) = \sin(100)$, again by subtracting off multiples of $2\pi$ from 100. $-\pi < 100 - 32\pi < \pi$, so we will use $t = 100 - 32\pi$.

$\left|R_n(t)\right| \le \frac{1}{(n+1)!} \cdot t^{n+1}$, and we require this to be less than $10^{-6}$. Solving $\frac{t^{n+1}}{(n+1)!} \le 10^{-6}$ with a table, we find that we need $n$ to be at least 7. (Note for comparison that to evaluate sin(100) directly with this level of precision requires a 275th-degree Maclaurin polynomial.)

d. Suppose we want to evaluate sin($u$). We begin by replacing $u$ with a number $t$ between $-\pi$ and $\pi$ such that $\sin(u) = \sin(t)$. The periodicity of the sine function guarantees that we will be able to find such a number $t$, and the same goes for the cosine function. Now the maximum distance that $t$ can be from the origin is $\pi$. Therefore, the error for *any* $t$-value between $-\pi$ and $\pi$ is bounded by $\left|R_n(t)\right| \le \frac{1}{(n+1)!} \cdot \left|\pi\right|^{n+1}$. (We're using 1 for $M$ again because all derivatives of sine and cosine are bounded by 1.) For $t$-values close to 0, the actual error will be much less than this estimate; this is a worst-case scenario error estimate. If we use a 29th-degree polynomial, as suggested, we have $\left|R_{29}(t)\right| \le \frac{1}{30!} \cdot \pi^{30} \approx 3.1 \times 10^{-18}$, which is well within the required error tolerance. In fact, $\left|R_{28}(t)\right| \le \frac{1}{29!} \cdot \pi^{29} \approx 3.0 \times 10^{-17}$. However, the Lagrange error bound estimate for a 27th-degree Maclaurin polynomial is not quite within the required precision. So we need to use at least a 28th-degree polynomial. Since the sine polynomials have only odd-degree terms, we must err on the side of caution and use a 29th-degree polynomial in general.

27. Picking up from Problem 9, $\left|R_n(5)\right| \le \frac{M}{(n+1)!} \cdot (5-1)^{n+1}$, where $M$ is a bound for the $(n + 1)^{st}$ derivative on the interval [1, 5]. For $n \ge 1$, $f^{(n)}(x) = (-1)^{n+1} \cdot \frac{(n-1)!}{x^n}$, and the graph of $f^{(n)}$ approaches $y = 0$ monotonically as $x$ increases. Therefore the maximum of $f^{(n)}(x)$ must occur at $x = 1$. We can then take $M$ to be simply $\left((n+1)-1\right)!$. Now $\left|R_n(5)\right| \le \frac{n!}{(n+1)!} \cdot 4^{n+1} = \frac{4^{n+1}}{n+1}$. Unfortunately, this expression does not decrease with increasing $n$. We cannot find an $n$ for which the error will be guaranteed to be within the specified tolerance. A complete explanation for why our strategy has fallen apart will have to wait until we talk about intervals of convergence of Taylor series in a later section (at which point we will know that it was ridiculous to even attempt this problem). For now, we will have to be content with a graphical answer. The Taylor polynomials for $f(x) = \ln(x)$ appear to be a good fit for the function only within about 1 unit of $x = 1$. For $x$-values more than 1 unit away from $x = 1$, the graphs of the Taylor polynomials diverge sharply from the graph of $f$. The graphs suggest that we cannot use a Taylor polynomial centered at $x = 1$ to approximate ln(5), and indeed that is the case.

28. a. We will use the Maclaurin polynomial for $f(x) = e^x$ with $x = 0.001$. Our desired error is less than $10^{-10}$. By the Lagrange error bound, $\left|R_n(0.001)\right| \le \frac{M}{(n+1)!}(0.001)^{n+1}$. We require, then, that $\frac{M}{(n+1)!}(0.001)^{n+1} \le 10^{-10}$, where $M$ is a bound on the $(n + 1)^{st}$ derivative of $e^x$ – namely $e^x$. We're working with $e^x$ on a sub-interval of [0, 1], so for all $t$-values in the interval, $e^t \le e^1 < 3$. This means we can use 3 for $M$. (We can push it lower since $e^{0.001}$ is much smaller than $e^1$, but it turns out that it will make no difference in the answer to the question.)

Using a table of values to solve $\frac{3}{(n+1)!} \cdot 0.001^{n+1} \le 10^{-10}$, we find that $n$ must be at least 3.

A third-degree Maclaurin polynomial is required to obtain the desired accuracy for computing $e^{0.001}$.

b. Now $x = 14$. We require that $\frac{M}{(n+1)!}\cdot 14^{n+1} \le 10^{-10}$. A suitable $M$-value is $3^{14}$ (see Problem 24). Solving $\frac{3^{14}}{(n+1)!}\cdot 14^{n+1} \le 10^{-10}$ with a table, we find that we must take $n$ to be at least 66. We need a $66^{\text{th}}$-degree Maclaurin polynomial to obtain the desired accuracy for computing $e^{14}$.

c. $\left|R_3(14)\right| \le \frac{3^{14}}{4!}\cdot 14^4 \approx 7.7\times 10^9$; the error is stupendously large in this situation.

d. $\left|R_{66}(0.001)\right| \le \frac{3}{67!}\cdot 0.001^{67} \approx 8\times 10^{-296}$, a *ridiculous* level of precision—one that the calculator cannot actually effectively use.

29. a. We assume that $0 < e < 3$ (which can be proved later). We further assume (for later contradiction) that $e = \frac{p}{q}$, where $p$ and $q$ are positive integers. Let $n$ be an integer greater than $q$ and greater than 3, and let $f(x) = e^x$. Then $f(x) \approx P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$. By Taylor's Theorem, $f(x) = P_n(x) + R_n(x)$, or

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + R_n(x).$$

Now plug in $x = 1$. This gives

$$e = \frac{p}{q} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + R_n(1)$$

as desired.

b. Multiplying through by $n!$ gives

$$\frac{p}{q}\cdot n! = n!\cdot\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) + n!\cdot R_n(1).$$

Since $n$ is greater than $q$, $n!$ has $q$ as a factor. Therefore $\frac{p}{q}\cdot n!$ must be an integer; the denominator $q$ will cancel with a factor of $n!$.

On the right side, $n!\cdot\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) = n! + n! + \frac{n!}{2!} + \frac{n!}{3!} + \cdots + \frac{n!}{n!}$ is also an integer. This means that Equation (3) has the form

$$\text{Integer} = \text{Integer} + n!\cdot R_n(1).$$

For this equation to be true, $n!\cdot R_n(1)$ must also be an integer. If we can show that $n!\cdot R_n(1)$ is *not* on integer, then we will have a contradiction. In that case we reject the assumption that $e$ can be expressed as a fraction of integers *p/q*. The conclusion will be that $e$ is irrational.

c. Using the Lagrange error bound, $\left|R_n(1)\right| \le \frac{M}{(n+1)!}\cdot(1-0)^{n+1} = \frac{M}{(n+1)!}$. Since $f(x) = e^x$ is increasing on [0, 1], we can use $e^1 = e$ as our $M$-value. We know / have assumed that $e < 3$, so 3 is also a suitable $M$-value. This gives $\left|R_n(1)\right| \le \frac{3}{(n+1)!}$, as desired.

d. Multiplying the inequality from part (c) by $n!$ gives $\left|n!\cdot R_n(1)\right| \le \frac{n!\cdot 3}{(n+1)!} = \frac{3}{n+1}$. (The $n!$ can slide into the absolute value bars because it is positive.) But recall that $n$ was chosen to be larger than 3. Therefore the right side of this inequality must be a number between 0 and 1, and it follows that so must $\left|n!\cdot R_n(1)\right|$. (The absolute value bars are proving to be pretty handy here. Without them, we could only conclude that $n!\cdot R_n(1)$ was less than 1. It could still be an integer… just a negative one. But because the absolute value bars trap the quantity in the non-negative world, we can conclude that $\left|n!\cdot R_n(1)\right|$ is not an integer.) Since $\left|n!\cdot R_n(1)\right|$ is not an integer, $n!\cdot R_n(1)$ is not either. Based on the comments in part (b), this completes the proof that $e$ is irrational.

30. a. $f(x) = P_n(x) + R_n(x) \Rightarrow f^{(n+1)}(x) = P_n^{(n+1)}(x) + R_n^{(n+1)}(x)$

However, $P_n(x)$ is an $n^{\text{th}}$-degree polynomial, so $P_n^{(n+1)}(x) = 0$. Substituting 0 for $P_n(x)$ in the previous equation gives $f^{(n+1)}(x) = R_n^{(n+1)}(x)$. For a number $t$ in $[a, x]$, we have $f^{(n+1)}(t) = R_n^{(n+1)}(t)$, as desired.

b. By hypothesis, $f^{(n+1)}(t)$ is bounded by $M$ for $t \in [a, x]$. This means that $-M \le f^{(n+1)}(t) \le M$. Substituting $R_n^{(n+1)}(t)$ for $f^{(n+1)}(t)$ as justified by part (a) gives $-M \le R_n^{(n+1)}(t) \le M$.

c. It will become clear in the integration why it is important to know $R_n^{(n)}(a)$; for now let's determine its value. Differentiating $f(x) = P_n(x) + R_n(x)$ $n$ times gives $f^{(n)}(x) = P_n^{(n)}(x) + R_n^{(n)}(x)$. We plug in $a$ to obtain $f^{(n)}(a) = P_n^{(n)}(a) + R_n^{(n)}(a)$. But by the definition of a Taylor polynomial, $f^{(n)}(a) = P_n^{(n)}(a)$. Therefore $R_n^{(n)}(a) = 0$.
Now for the integration…

$$\begin{array}{ccccc}
-M & \le & R_n^{(n+1)}(t) & \le & M \\
\int_a^x -M\,dt & \le & \int_a^x R_n^{(n+1)}(t)\,dt & \le & \int_a^x M\,dt \\
-M\,t\Big|_a^x & \le & R_n^{(n)}(t)\Big|_a^x & \le & Mt\Big|_a^x \\
-M(x-a) & \le & R_n^{(n)}(x) - R_n^{(n)}(a) & \le & M(x-a) \\
-M(x-a) & \le & R_n^{(n)}(x) & \le & M(x-a)
\end{array}$$

d. Replace $x$ with the dummy variable $t$ in the result of part (c) and integrate again. Note that for any $k$ such that $0 \le k \le n$, $R_n^{(k)}(a) = 0$ since the $k^{\text{th}}$ derivative of $f$ and $P_n$ will agree perfectly at the center. (This is a generalization of the observation in part (c) about $R_n^{(n)}(a)$.)

$$\begin{array}{ccccc}
-M(t-a) & \le & R_n^{(n)}(t) & \le & M(t-a) \\
\int_a^x -M(t-a)\,dt & \le & \int_a^x R_n^{(n)}(t)\,dt & \le & \int_a^x M(t-a)\,dt \\
-M \cdot \frac{(t-a)^2}{2}\Big|_a^x & \le & R_n^{(n-1)}(t)\Big|_a^x & \le & M \cdot \frac{(t-a)^2}{2}\Big|_a^x \\
-M \cdot \frac{(t-a)^2}{2} & \le & R_n^{(n-1)}(x) - R_n^{(n-1)}(a) & \le & M \cdot \frac{(t-a)^2}{2} \\
-M \cdot \frac{(t-a)^2}{2} & \le & R_n^{(n-1)}(x) & \le & M \cdot \frac{(t-a)^2}{2}
\end{array}$$

Only $n - 1$ integrations left to go! We omit the details here (which can be filled in using mathematical induction), but the pattern should be clear. With each integration, $R_n^{(k)}(x)$ becomes $R_n^{(k-1)}(x)$. Similarly, the factors $\frac{(x-a)^n}{n!}$ become $\frac{(x-a)^{n+1}}{(n+1)!}$. Therefore, we will ultimately obtain $\frac{-M}{(n+1)!}(x-a)^{n+1} \le R_n(x) \le \frac{M}{(n+1)!}(x-a)^{n+1}$.

e. The result of part (d) is exactly the same as the statement $\left|R_n(x)\right| \le \frac{M}{(n+1)!}(x-a)^{n+1}$. Implicit in our argument so far has been the assumption that $x > a$. This need not be the case, though. (Some house-keeping details in the proof are necessary to take care of the $x < a$ case, but they are just details.) To make sure our bounding value is actually positive, we replace $(x - a)$ with $|x - a|$ so that $\left|R_n(x)\right| \le \frac{M}{(n+1)!}\left|x-a\right|^{n+1}$. This completes the argument.

## Section 6

1. $a_n = \frac{n^2}{2^n}$, so $a_{n+1} = \frac{(n+1)^2}{2^{n+1}}$. $\lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{1}{2} = \frac{1}{2} < 1$. The series converges.

2. $a_n = \frac{n}{4^n}$, so $a_{n+1} = \frac{n+1}{4^{n+1}}$. $\lim_{n\to\infty} \frac{n+1}{4^{n+1}} \cdot \frac{4^n}{n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{4^n}{4^{n+1}} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{1}{4} = \frac{1}{4} < 1$. The series converges.

3. $\lim\limits_{n\to\infty}\frac{2}{3^{n+1}}\cdot\frac{3^n}{2}=\lim\limits_{n\to\infty}\frac{3^n}{3^{n+1}}=\frac{1}{3}<1$. The series converges.

4. $\lim\limits_{n\to\infty}\frac{1}{(n+1)^2}\cdot\frac{n^2}{1}=\lim\limits_{n\to\infty}\frac{n^2}{(n+1)^2}=1$. The ratio test is inconclusive.

5. $\lim\limits_{n\to\infty}\frac{(n+1)!}{(2n+2)!}\cdot\frac{(2n)!}{n!}=\lim\limits_{n\to\infty}\frac{(n+1)!}{n!}\cdot\frac{(2n)!}{(2n+2)!}=\lim\limits_{n\to\infty}\frac{(n+1)}{(2n+2)(2n+1)}=0<1$. The series converges.

6. $\lim\limits_{n\to\infty}\frac{[(n+1)!]^2}{(2n+2)!}\cdot\frac{(2n)!}{(n!)^2}=\lim\limits_{n\to\infty}\frac{(2n)!}{(2n+2)!}\cdot\frac{(n+1)!}{n!}\cdot\frac{(n+1)!}{n!}=\lim\limits_{n\to\infty}\frac{(n+1)(n+1)}{(2n+2)(2n+1)}=\frac{1}{4}<1$. The series converges.

7. $\lim\limits_{n\to\infty}\frac{n+2}{(n+1)^3}\cdot\frac{n^3}{n+1}=\lim\limits_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{n^3}{(n+1)^3}=1$. The ratio test is inconclusive.

8. $\lim\limits_{n\to\infty}\frac{1}{(2n+3)!}\cdot\frac{(2n+1)!}{1}=\lim\limits_{n\to\infty}\frac{(2n+1)!}{(2n+3)!}=\lim\limits_{n\to\infty}\frac{1}{(2n+3)(2n+2)}=0<1$. The series converges.

9. $\lim\limits_{n\to\infty}\frac{(n+1)\cdot 4^{n+1}}{5^{n+1}}\cdot\frac{5^n}{n\cdot 4^n}=\lim\limits_{n\to\infty}\frac{n+1}{n}\cdot\frac{4^{n+1}}{4^n}\cdot\frac{5^n}{5^{n+1}}=\lim\limits_{n\to\infty}\frac{n+1}{n}\cdot\frac{4}{5}=\frac{4}{5}<1$. The series converges.

10. $\lim\limits_{n\to\infty}\frac{(n+2)!}{(n+1)\cdot 3^{n+1}}\cdot\frac{n3^n}{(n+1)!}=\lim\limits_{n\to\infty}\frac{n}{n+1}\cdot\frac{(n+2)!}{(n+1)!}\cdot\frac{3^n}{3^{n+1}}=\lim\limits_{n\to\infty}\frac{n}{n+1}\cdot(n+2)\cdot\frac{1}{3}=\infty>1$ The series diverges.

11. $\lim\limits_{n\to\infty}\frac{4^{n+1}}{(2n+1)!}\cdot\frac{(2n-1)!}{4^n}=\lim\limits_{n\to\infty}\frac{4^{n+1}}{4^n}\cdot\frac{(2n-1)!}{(2n+1)!}=\lim\limits_{n\to\infty}4\cdot\frac{1}{(2n+1)(2n)}=0<1$. The series converges.

12. $\lim\limits_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}=\lim\limits_{n\to\infty}\frac{n!}{(n+1)!}\cdot\frac{(n+1)^{n+1}}{n^n}=\lim\limits_{n\to\infty}\frac{1}{n+1}\cdot\frac{(n+1)\cdot(n+1)^n}{n^n}=\lim\limits_{n\to\infty}\frac{\cancel{n+1}}{\cancel{n+1}}\cdot\frac{(n+1)^n}{n^n}=\lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^n=\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e>1$. The series diverges.

Particular approaches to problems 13-17 may vary. The solutions presented here are not necessarily unique.

13. $a_n=\frac{n+2}{n+6}$. $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\frac{n+2}{n+6}=1\neq 0$. The series diverges by the $n^{\text{th}}$ term test.

14. $a_n=\frac{3^n}{n!}$. $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n}=\lim\limits_{n\to\infty}\frac{3^{n+1}}{3^n}\cdot\frac{n!}{(n+1)!}=\lim\limits_{n\to\infty}\frac{3}{n+1}=0<1$. The series converges by the ratio test.

15. The series is geometric with $|r|=\left|\frac{-1}{3}\right|=\frac{1}{3}<1$. The series converges by the geometries series test.

16. This is the harmonic series. It diverges.

17. $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\frac{n!}{2n^5}=\infty\neq 0$. The series diverges by the $n^{\text{th}}$ term test.

18. a. $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{1}{4}<1$. The series converges by the ratio test.

    b. $\lim\limits_{n\to\infty}\frac{1/a_{n+1}}{1/a_n}=\lim\limits_{n\to\infty}\frac{a_n}{a_{n+1}}=4>1$. The series diverges by the ratio test.

    c. $\lim\limits_{n\to\infty}\frac{(n+1)a_{n+1}}{na_n}=\lim\limits_{n\to\infty}\frac{n+1}{n}\cdot\frac{a_{n+1}}{a_n}=1\cdot\frac{1}{4}<1$. The series converges by the ratio test.

    d. $\lim\limits_{n\to\infty}\frac{(n+1)^3a_{n+1}}{n^3a_n}=\lim\limits_{n\to\infty}\frac{(n+1)^3}{n^3}\cdot\frac{a_{n+1}}{a_n}=1\cdot\frac{1}{4}<1$. The series converges by the ratio test.

    e. $\lim\limits_{n\to\infty}\frac{a_{n+1}/(n+1)}{a_n/n}=\lim\limits_{n\to\infty}\frac{n}{n+1}\cdot\frac{a_{n+1}}{a_n}=1\cdot\frac{1}{4}<1$. The series converges by the ratio test.

    f. $\lim\limits_{n\to\infty}\frac{(a_{n+1})^2}{(a_n)^2}=\lim\limits_{n\to\infty}\frac{a_{n+1}\cdot a_{n+1}}{a_n\cdot a_n}=\frac{1}{16}<1$. The series converges by the ratio test.

    g. $\lim\limits_{n\to\infty}\frac{2^{n+1}a_{n+1}}{2^n a_n}=\lim\limits_{n\to\infty}\frac{2^{n+1}}{2^n}\cdot\frac{a_{n+1}}{a_n}=2\cdot\frac{1}{4}<1$. The series converges by the ratio test.

    h. $\lim\limits_{n\to\infty}\frac{5^{n+1}a_{n+1}}{5^n a_n}=\lim\limits_{n\to\infty}\frac{5^{n+1}}{5^n}\cdot\frac{a_{n+1}}{a_n}=5\cdot\frac{1}{4}>1$. The series diverges by the ratio test.

19. $\lim\limits_{n\to\infty}\frac{n+1}{r^{n+1}}\cdot\frac{r^n}{n}=\lim\limits_{n\to\infty}\frac{n+1}{n}\cdot\frac{r^n}{r^{n+1}}=1\cdot\frac{1}{r}=\frac{1}{r}$. If $r>1$, as given, then $1/r<1$. In this case, the series converges by the ratio test. (Further note that if $0<r<1$, then $1/r>1$ and the series will diverge by the ratio test. If $r=1$, the series diverges by the $n^{\text{th}}$ term test. Negative $r$-values will have to wait until Section 8.)

20. a. This <u>is</u> a power series; center is $x=-2$.
    b. This is <u>not</u> a power series.
    c. This <u>is</u> a power series; center is $x=3$.
    d. This is <u>not</u> a power series.

e. This is a power series; center is $x$ = -1.
f. This is not a power series.

21. a. The center of the series is 0, and the series converges at $x$ = -2, which is two units away. The smallest possible radius of convergence is 2.
b. The series diverges at $x$ = 5, which is five units from the center. The largest possible radius of convergence is 5.
c. Definitely converges (within 2 units of the center): -1, 0, 1
Definitely diverges (more than 5 units from the center): -8
Cannot be determined (between 2 and 5 units, inclusive, from the center): -5, 2, 4
22. a. The center of the series is 3, and the series converges at $x$ = 0, which is three units away. The smallest possible radius of convergence is 3.
b. The series diverges at $x$ = -2, five units from the center. The largest possible radius of convergence is 5.
c. Definitely converges (within 3 units of the center): 2, 3, 5
Definitely diverges (more than 5 units from the center): -3, 9
Cannot be determined (between 3 and 5 units, inclusive, from the center): -1, 6, 8
23. a. No. [-5, 5] is symmetric about 0, but the center of the series is -1. The interval of convergence of the series must be symmetric about the center (give or take the endpoints).
b. No. This interval is not symmetric about $x$ = -1. Also, it doesn't even contain $x$ = 5 in the interval of convergence!
c. Yes. This interval is symmetric about $x$ = -1 (give or take the endpoints) and contains $x$ = 5.
d. Yes. This interval is symmetric about $x$ = -1 (give or take the endpoints) and contains $x$ = 5.
24. The new series is the derivative of the original series and therefore has the same radius of convergence. $R$ = 5.
25. No! The set of $x$-values for which a power series converges is a single interval. At best, the situation described here is of a series that converges for two different intervals separated by $x$ = 6. That can't happen with a power series.
26. This power series is geometric with $r=\frac{x}{5}$. It converges if $\left|\frac{x}{5}\right|<1$, equivalently $|x|<5$. The radius of convergence is 5.
27. $a_n=\frac{(x-2)^n}{n\cdot 2^n}$. $\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)\cdot 2^{n+1}}\cdot\frac{n\cdot 2^n}{(x-2)^n}\right|=\lim_{n\to\infty}\left|\frac{n}{n+1}\cdot\frac{2^n}{2^{n+1}}\cdot\frac{(x-2)^{n+1}}{(x-2)^n}\right|=\lim_{n\to\infty}\left|\frac{n}{n+1}\cdot\frac{1}{2}\cdot(x-2)\right|=\frac{|x-2|}{2}$. We require $\frac{|x-2|}{2}<1$ for convergence, or equivalently $|x-2|<2$. The radius of convergence is 2.
28. $\lim_{n\to\infty}\left|\frac{(4x)^{n+1}}{(4x)^n}\right|=|4x|=4|x|$. $4|x|<1\Rightarrow|x|<\frac{1}{4}$. The radius of convergence is 1/4.
29. $\lim_{n\to\infty}\left|\frac{(x+4)^{n+1}}{(n+1)\cdot 3^{n+1}}\cdot\frac{n\cdot 3^n}{(x+4)^n}\right|=\lim_{n\to\infty}\left|\frac{n}{n+1}\cdot\frac{3^n}{3^{n+1}}\cdot\frac{(x+4)^{n+1}}{(x+4)^n}\right|=\lim_{n\to\infty}\left|\frac{n}{n+1}\cdot\frac{1}{3}\cdot(x+4)\right|=\frac{|x+4|}{3}$. $\frac{|x+4|}{3}<1\Rightarrow|x+4|<3$. The radius of convergence is 3.
30. $\lim_{n\to\infty}\left|\frac{3^{n+1}\cdot(x+1)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{3^n\cdot(x+1)^n}\right|=\lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\cdot\frac{3^{n+1}}{3^n}\cdot\frac{(x+1)^{n+1}}{(x+1)^n}\right|=\lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\cdot 3\cdot(x+1)\right|=3|x+1|$. $3\cdot|x+1|<1\Rightarrow|x+1|<\frac{1}{3}$. The radius of convergence is 1/3.
31. $\lim_{n\to\infty}\left|\frac{(n+1)!\cdot x^{n+1}}{(n+1)^3\cdot 4^{n+1}}\cdot\frac{n^3\cdot 4^n}{n!\cdot x^n}\right|=\lim_{n\to\infty}\left|\cancel{\frac{n^3}{(n+1)^3}}\cdot\frac{4^n}{4^{n+1}}\cdot\frac{(n+1)!}{n!}\cdot\frac{x^{n+1}}{x^n}\right|=\lim_{n\to\infty}\left|\frac{1}{4}\cdot(n+1)\cdot x\right|=\infty>1$. The radius of convergence of this series is 0. The series converges only at its center.
32. $\lim_{n\to\infty}\left|\frac{(n+2)\cdot(x-5)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(n+1)\cdot(x-5)^n}\right|=\lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\cdot\frac{(n+2)}{(n+1)}\cdot\frac{(x-5)^{n+1}}{(x-5)^n}\right|=|x-5|$. We require that $|x-5|<1$, so the radius of convergence is 1.
33. This is a geometric series with $r=\frac{x+5}{3}$. It converges as long as $\left|\frac{x+5}{3}\right|<1$, equivalently, $|x+5|<3$. The radius of convergence of the series is 3.

34. $\lim_{n\to\infty}\left|\frac{2^{n+1}\cdot(x-1)^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n\cdot(x-1)^n}\right|=\lim_{n\to\infty}\left|\frac{2^{n+1}}{2^n}\cdot\frac{n!}{(n+1)!}\cdot\frac{(x-1)^{n+1}}{(x-1)^n}\right|=\lim_{n\to\infty}\left|2\cdot\frac{1}{n+1}\cdot(x-1)\right|=0<1$. This series converges for all $x$. Its radius of convergence is $\infty$.
35. $\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\cdot x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(-1)^n\cdot x^n}\right|=\lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\cdot\frac{x^{n+1}}{x^n}\right|=\lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\cdot x\right|=|x|$. We require that $|x|<1$, so the radius of convergence is 1.
36. Note first that $\cos(n\pi)=(-1)^n$. When we take the absolute values, this factor will be irrelevant. $\lim_{n\to\infty}\left|\frac{\cos((n+1)\pi)\cdot(x+2)^{n+1}}{3^{n+1}}\cdot\frac{3^n}{\cos(n\pi)\cdot(x+2)^n}\right|=\lim_{n\to\infty}\left|\frac{3^n}{3^{n+1}}\cdot\frac{(x+2)^{n+1}}{(x+2)^n}\right|=\left|\frac{x+2}{3}\right|$. $\left|\frac{x+2}{3}\right|<1\Rightarrow|x+2|<3$. The radius of convergence is 3.
37. $\lim_{n\to\infty}\left|\frac{(-1)^{n+2}\cdot(x-4)^{2n+2}}{4n+4}\cdot\frac{4n}{(-1)^{n+1}\cdot(x-4)^{2n}}\right|=\lim_{n\to\infty}\left|\frac{4n}{4n+4}\cdot\frac{(x-4)^{2n+2}}{(x-4)^{2n}}\right|=\lim_{n\to\infty}\left|\frac{4n}{4n+4}\cdot(x-4)^2\right|=(x-4)^2$. $(x-4)^2<1\Rightarrow|x-4|<1$. The radius of convergence is 1.
38. $\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|=\lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\cdot\frac{x^{n+1}}{x^n}\right|=\lim_{n\to\infty}\left|\frac{x}{n+1}\right|=0<1$. This series converges for all $x$. The radius of convergence is $\infty$.
39. $\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{(-1)^n x^{2n+1}}\right|=\lim_{n\to\infty}\left|\frac{(2n+1)!}{(2n+3)!}\cdot\frac{x^{2n+3}}{x^{2n+1}}\right|=\lim_{n\to\infty}\left|\frac{1}{(2n+3)(2n+2)}\cdot x^2\right|=0<1$. This series converges for all $x$. The radius of convergence is $\infty$.
40. $\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{(-1)^n x^{2n}}\right|=\lim_{n\to\infty}\left|\frac{(2n)!}{(2n+2)!}\cdot\frac{x^{2n+2}}{x^{2n}}\right|=\lim_{n\to\infty}\left|\frac{1}{(2n+2)(2n+1)}\cdot x^2\right|=0<1$. This series converges for all $x$. The radius of convergence is $\infty$.
41. The graph of $y=f(5x)$ is like that of $y=f(x)$, except that it has been compressed horizontally by a factor of 5. The radius of convergence of the power series will be similarly compressed. The radius is $15\div 5=3$.
42. Because the radius of convergence of the first series is $R$, we know from the ratio test (applied to the first series) that $\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right|<1$ whenever $|x-a|<R$. Now we apply the ratio test to the second series: $\lim_{n\to\infty}\left|\frac{(n+1)c_{n+1}(x-a)^{n+1}}{nc_n(x-a)^n}\right|=\lim_{n\to\infty}\left|\cancel{\frac{n+1}{n}}\cdot\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right|=\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right|$. This last limit is still less than 1 when $|x-a|<R$. The radius of convergence of the second series is also $R$.
43. a. $\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\frac{n}{3^n}}=\lim_{n\to\infty}\frac{\sqrt[n]{n}}{\sqrt[n]{3^n}}=\lim_{n\to\infty}\frac{\sqrt[n]{n}}{3}=\frac{1}{3}<1$. This series converges.
    b. $\lim_{n\to\infty}\sqrt[n]{\frac{2}{n^3}}=\lim_{n\to\infty}\frac{\sqrt[n]{2}}{\left(\sqrt[n]{n}\right)^3}=\frac{1}{1}=1$. The root test is inconclusive.
    c. $\lim_{n\to\infty}\sqrt[n]{\left(\frac{2n+1}{n}\right)^n}=\lim_{n\to\infty}\frac{2n+1}{n}=2>1$. This series diverges.
    d. $\lim_{n\to\infty}\sqrt[n]{\frac{1}{n^n}}=\lim_{n\to\infty}\frac{1}{n}=0<1$. This series converges.
44. a. $\sum_{n=1}^{\infty}\left(\frac{3}{n}-\frac{3}{n+1}\right)=\left(\frac{3}{1}-\frac{3}{2}\right)+\left(\frac{3}{2}-\frac{3}{3}\right)+\left(\frac{3}{3}-\frac{3}{4}\right)+\cdots$. This series is telescoping. The general partial sum is given by $s_n=3-\frac{3}{n+1}$, which converges to 3 as $n\to\infty$.
    b. $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{2^{n+1}}{(3n+3)!}\cdot\frac{(3n)!}{2^n}=\lim_{n\to\infty}\frac{2}{(3n+3)(3n+2)(3n+1)}=0<1$. This series converges by the ratio test.
    c. We can use either the ratio test or our new friend from Problem 43: the root test. I will use the latter. $\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\frac{1}{e^n}}=\lim_{n\to\infty}\frac{1}{e}=\frac{1}{e}<1$. This series converges by the root test.
    d. $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{3^n+1}{3^n}=1\neq 0$. This series diverges by the $n^{\text{th}}$ term test.
    e. This series is geometric with $|r|=\frac{3}{4}<1$. The series converges by the geometric series test.

f. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n}{n} = \infty \neq 0$. This series diverges by the $n^{\text{th}}$ term test. (The ratio test could also be used.)

45. a. In this series we have $a_n = \frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{2\cdot4\cdot6\cdot\cdots\cdot(2n)}$, so $a_{n+1} = \frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)(2n+1)}{2\cdot4\cdot6\cdot\cdots\cdot(2n)(2n+2)}$. It follows that $\frac{a_n}{a_{n+1}} = \frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{2\cdot4\cdot6\cdot\cdots\cdot(2n)} \cdot \frac{2\cdot4\cdot6\cdot\cdots\cdot(2n)(2n+2)}{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)(2n+1)} = \frac{\cancel{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}}{\cancel{2\cdot4\cdot6\cdot\cdots\cdot(2n)}} \cdot \frac{\cancel{2\cdot4\cdot6\cdot\cdots\cdot(2n)}\,(2n+2)}{\cancel{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}\,(2n+1)} = \frac{2n+2}{2n+1}$. Now we are ready to evaluate the limit.

$\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right] = \lim_{n\to\infty}\left[n\left(\frac{2n+2}{2n+1}-1\right)\right] = \lim_{n\to\infty}\left[n\cdot\frac{1}{2n+1}\right] = \frac{1}{2} < 1$. This series diverges.

b. $a_n = \frac{2\cdot4\cdot6\cdot\cdots\cdot(2n)}{5\cdot7\cdot9\cdot\cdots\cdot(2n+3)}$ and $a_{n+1} = \frac{2\cdot4\cdot6\cdot\cdots\cdot(2n)(2n+2)}{5\cdot7\cdot9\cdot\cdots\cdot(2n+3)(2n+5)}$, so $\frac{a_n}{a_{n+1}} = \frac{\cancel{2\cdot4\cdot6\cdot\cdots\cdot(2n)}}{\cancel{5\cdot7\cdot9\cdot\cdots\cdot(2n+3)}} \cdot \frac{\cancel{5\cdot7\cdot9\cdot\cdots\cdot(2n+3)}\,(2n+5)}{\cancel{2\cdot4\cdot6\cdot\cdots\cdot(2n)}\,(2n+2)} = \frac{2n+5}{2n+2}$.

$\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right] = \lim_{n\to\infty}\left[n\left(\frac{2n+5}{2n+2}-1\right)\right] = \lim_{n\to\infty}\left[n\cdot\frac{3}{2n+2}\right] = \frac{3}{2} > 1$. This series converges.

c. $a_n = \frac{2\cdot5\cdot8\cdot\cdots\cdot(3n-1)}{4\cdot7\cdot10\cdot\cdots\cdot(3n+1)}$ and $a_{n+1} = \frac{2\cdot5\cdot8\cdot\cdots\cdot(3n-1)(3n+2)}{4\cdot7\cdot10\cdot\cdots\cdot(3n+1)(3n+4)}$, so $\frac{a_n}{a_{n+1}} = \frac{2\cdot5\cdot8\cdot\cdots\cdot(3n-1)}{4\cdot7\cdot10\cdot\cdots\cdot(3n+1)} \cdot \frac{4\cdot7\cdot10\cdot\cdots\cdot(3n+1)(3n+4)}{2\cdot5\cdot8\cdot\cdots\cdot(3n-1)(3n+2)} = \frac{3n+4}{3n+2}$.

$\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right] = \lim_{n\to\infty}\left[n\left(\frac{3n+4}{3n+2}-1\right)\right] = \lim_{n\to\infty}\left[n\cdot\frac{2}{3n+2}\right] = \frac{2}{3} < 1$. This series diverges.

d. $a_n = \left[\frac{2\cdot4\cdot6\cdot\cdots\cdot(2n)}{5\cdot7\cdot9\cdot\cdots\cdot(2n+3)}\right]^{2/3}$ and $a_{n+1} = \left[\frac{2\cdot4\cdot6\cdot\cdots\cdot(2n)(2n+2)}{5\cdot7\cdot9\cdot\cdots\cdot(2n+3)(2n+5)}\right]^{2/3}$, so $\frac{a_n}{a_{n+1}} = \left[\frac{2\cdot4\cdot6\cdot\cdots\cdot(2n)}{5\cdot7\cdot9\cdot\cdots\cdot(2n+3)} \cdot \frac{5\cdot7\cdot9\cdot\cdots\cdot(2n+3)(2n+5)}{2\cdot4\cdot6\cdot\cdots\cdot(2n)(2n+2)}\right]^{2/3}$ or $\frac{a_n}{a_{n+1}} = \left(\frac{2n+5}{2n+2}\right)^{2/3}$. Now for the limit. $\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right] = \lim_{n\to\infty}\left[n\left(\left(\frac{2n+5}{2n+2}\right)^{2/3}-1\right)\right] = 1$. The simplest way to show that this limit equal 1 is to use a CAS, but I guess that's not why you look in a solution manual. Okay. Here we go.

First note that the limit here has the indeterminate form $\infty\cdot 0$. Clearly $n$ goes to ∞. Furthermore, since $\frac{2n+5}{2n+2}$ goes to 1, $\left(\frac{2n+5}{2n+2}\right)^{2/3}-1$ goes to 0. If we rewrite the multiplication by $n$ as division by $1/n$, then we can use l'Hospital's rule along with a *lot* of algebra.

$$\lim_{n\to\infty}\left[n\left(\left(\frac{2n+5}{2n+2}\right)^{2/3}-1\right)\right] = \lim_{n\to\infty}\frac{\left(\frac{2n+5}{2n+2}\right)^{2/3}-1}{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\frac{\frac{2}{3}\left(\frac{2n+5}{2n+2}\right)^{-1/3}\cdot\frac{-6}{(2n+2)^2}}{\frac{-1}{n^2}}$$

$$= \lim_{n\to\infty}\frac{4n^2\cdot(2n+2)^{1/3}}{(2n+2)^2(2n+5)^{1/3}} = \lim_{n\to\infty}\frac{4n^2}{(2n+2)^{5/3}(2n+5)^{1/3}}$$

$$= \lim_{n\to\infty}\frac{4n^2}{\left((2n+2)^5(2n+5)\right)^{1/3}} = \lim_{n\to\infty}\frac{4n^2}{\left(64n^6+\text{lower order terms}\right)^{1/3}}$$

It should be clear that this last form of the limit is equivalent to $\lim_{n\to\infty}\frac{4n^2}{4n^2}$, the lower order terms being inconsequential as $n\to\infty$. Hence, we have the desired result, that the limit is 1. Unfortunately after all that effort, since the limit is 1, the Raabe test is inconclusive for this series.

46. $a_n = \frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{2\cdot5\cdot8\cdot\cdots\cdot(3n-1)}$ and $a_{n+1} = \frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)(2n+1)}{2\cdot5\cdot8\cdot\cdots\cdot(3n-1)(3n+2)}$, so $\frac{a_{n+1}}{a_n} = \frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)(2n+1)}{2\cdot5\cdot8\cdot\cdots\cdot(3n-1)(3n+2)} \cdot \frac{2\cdot5\cdot8\cdot\cdots\cdot(3n-1)}{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)} = \frac{2n+1}{3n+2}$.

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{2n+1}{3n+2} = \frac{2}{3} < 1$. This series converges by the ratio test.

47. a. We start by ignoring the 2. Long-term, it will not matter. We apply the ratio test to the absolute value of the general term of the series.

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+2}\cdot(2n-1)!!\cdot(x-4)^{n+1}}{(n+1)!\cdot 2^{3n+2}} \cdot \frac{n!\cdot 2^{3n-1}}{(-1)^n\cdot(2n-3)!!\cdot(x-4)^n}\right| = \left|\frac{(2n-1)!!}{(2n-3)!!}\cdot\frac{n!}{(n+1)!}\cdot\frac{2^{3n-1}}{2^{3n+2}}\cdot\frac{(x-4)^{n+1}}{(x-4)^n}\right| = \left|(2n-1)\cdot\frac{1}{n+1}\cdot\frac{1}{8}\cdot(x-4)\right|$

And now for the limit: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2n-1}{n+1}\cdot\frac{x-4}{8}\right| = \frac{|x-4|}{4}$. $\frac{|x-4|}{4} < 1 \Rightarrow |x-4| < 4$. The radius of convergence for this power series is 4.

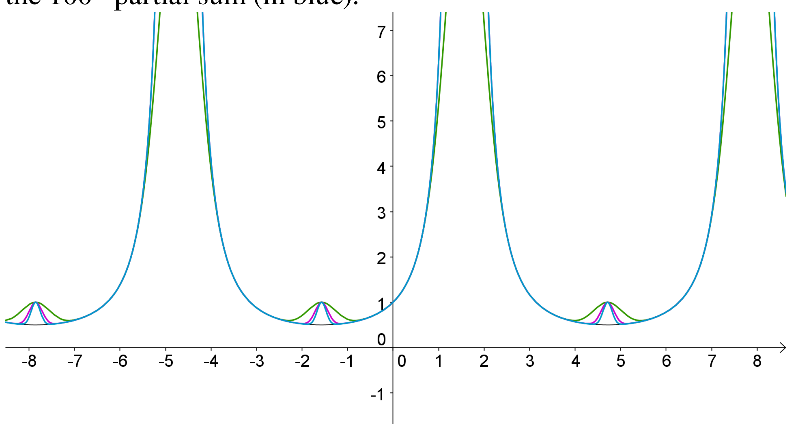
b. $a_n = \frac{(2n-3)!!\cdot 4^n}{n!\cdot 2^{3n-1}}$ and $a_{n+1} = \frac{(2n-1)!!\cdot 4^{n+1}}{(n+1)!\cdot 2^{3n+2}}$, so $\frac{a_n}{a_{n+1}} = \frac{(2n-3)!!\cdot 4^n}{n!\cdot 2^{3n-1}} \cdot \frac{(n+1)!\cdot 2^{3n+2}}{(2n-1)!!\cdot 4^{n+1}} = \frac{(2n-3)!!}{(2n-1)!!} \cdot \frac{(n+1)!}{n!} \cdot \frac{4^n}{4^{n+1}} \cdot \frac{2^{3n+2}}{2^{3n-1}}$. Simplifying gives $\frac{a_{n+1}}{a_n} = \frac{1}{2n-1} \cdot (n+1) \cdot \frac{1}{4} \cdot 8 = \frac{n+1}{2n-1} \cdot 2 = \frac{2n+2}{2n-1}$. Now we are ready to evaluate the limit in the Raabe test: $\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right] = \lim_{n\to\infty}\left[n\left(\frac{2n+2}{2n-1}-1\right)\right] = \lim_{n\to\infty}\left[n\cdot\frac{3}{2n-1}\right] = \frac{3}{2} > 1$. By the Raabe test, this series converges.

c. If we ignore the alternating factor, then we have exactly the same general term as before. Since the series in part (b) converged, this series converges as well.

d. The power series converges for $0 \le x \le 8$.

48. This series is geometric with $r = \sin(x)$. It converges as long as $|r| = |\sin x| < 1$. Specifically, the series converges as long as $x \ne \frac{\pi}{2} + k\pi$ (where $k$ is an integer). The series does not converge on an interval. Instead it converges on infinitely many distinct intervals.

Below is a graph of $f(x) = \frac{1}{1-\sin(x)}$ (in black, only really visible at the bottom of each trough, where the graphs diverge), the 10th partial sum of the series (in green), the 50th partial sum (in purple), and the 100th partial sum (in blue).

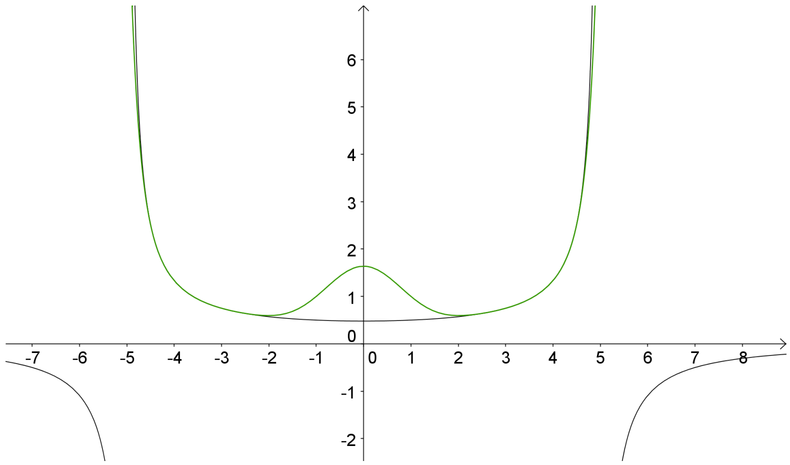



49. a The general term cannot be written, even by lots of algebra, in the form $c_n(x-a)^n$.

b. This series is, however, geometric with $a = 1$ and $r = \frac{x^2-13}{12}$. As long as $|r| < 1$, the series converges to $f(x) = \dfrac{1}{1-\frac{x^2-13}{12}} = \dfrac{12}{12-\left(x^2-13\right)} = \dfrac{12}{25-x^2}$.

c. The series converges when $\left|\frac{x^2-13}{12}\right| < 1$. $\left|\frac{x^2-13}{12}\right| < 1 \Rightarrow \left|x^2-13\right| < 12$ or $-12 < x^2 - 13 < 12$. Adding 13 gives $1 < x^2 < 25$. Square-rooting (and being careful of signs and such), we find that the series converges on the two separate intervals $-5 < x < -1$ and $1 < x < 5$.

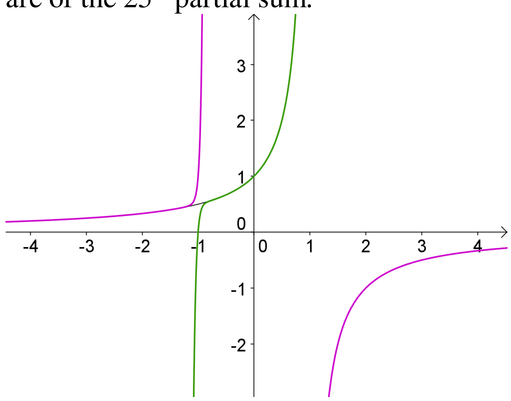
d. Here is a graph of the 10th partial sum (in green) along with $f(x)$ (in black).



50. a. $f(x)=\dfrac{1}{1-x}=\dfrac{1}{1-x}\cdot\dfrac{\frac{-1}{x}}{\frac{-1}{x}}=\dfrac{\frac{-1}{x}}{\frac{-1}{x}-x\cdot\frac{-1}{x}}=\dfrac{\frac{-1}{x}}{1-\frac{1}{x}}$

b. We have $a = -1/x$ and $r = 1/x$. Therefore $f(x)=\frac{-1}{x}-\frac{1}{x^2}-\frac{1}{x^3}-\cdots-\frac{1}{x^n}-\cdots=\sum_{n=1}^{\infty}\frac{-1}{x^n}$. The powers of $x$ are negative integers, not positive integers. This means that the series is not a power-series. Conceptually, it is not "polynomial-like."

c. The graph below shows the power series in green and the Laurent series in purple. Both graphs are of the 25th partial sum.



51. $y=f(x)=\frac{2x^3}{3!}+\frac{2x^4}{4!}+\frac{2x^5}{5!}+\cdots+\frac{2x^n}{n!}+\cdots$. It follows that $y'=\frac{6x^2}{3!}+\frac{8x^3}{4!}+\frac{10x^4}{5!}+\cdots+\frac{2n\cdot x^{n-1}}{n!}+\cdots$. Simplifying $y'$ gives $y'=\frac{2x^2}{2!}+\frac{2x^3}{3!}+\frac{2x^4}{4!}\cdots+\frac{2x^{n-1}}{(n-1)!}+\cdots$ or $y'=x^2+\frac{2x^3}{3!}+\frac{2x^4}{4!}\cdots+\frac{2x^{n-1}}{(n-1)!}+\cdots$. By substituting the power series for $y$, we see that the right side of the differential equation is $x^2+y=x^2+\frac{2x^3}{3!}+\frac{2x^4}{4!}+\frac{2x^5}{5!}+\cdots+\frac{2x^n}{n!}+\cdots$. Notice that this is identical to the power series version of the left side of the differential equation. This solution checks; the power series does solve the differential equation.

## Section 7

1. The series is $\sum_{n=1}^{\infty}\frac{1}{n^4}$ which is a $p$-series with $p = 4 > 1$. The series converges by the $p$-series test.

2. The series is $\sum_{n=1}^{\infty}\frac{1}{4^n}$. This is a convergent geometric series, but the directions say to use either the $p$-series test or integral test. Since the series is not a $p$-series, we're stuck with the integral test. What a bother. $f(x)=\frac{1}{4^x}=4^{-x}$ is positive, continuous, and decreasing for all $x$.

$\int_1^{\infty}4^{-x}\,dx=\lim_{b\to\infty}\int_1^b 4^{-x}\,dx=\lim_{b\to\infty}\left(\frac{-1}{\ln 4}\cdot 4^{-x}\right)\Big|_1^b=\lim_{b\to\infty}\left(\frac{-1}{4^b\ln 4}+\frac{1}{4\ln 4}\right)=\frac{1}{4\ln 4}$. Since this integral converges, the series converges by the integral test.

3. The series is $\sum_{n=1}^{\infty}\frac{n}{e^{n^2}}$. Let $f(x)=\frac{x}{e^{x^2}}$. This function is positive, continuous, and decreasing for all $x\geq 1$. $\int_1^{\infty}\frac{x}{e^{x^2}}dx=\lim_{b\to\infty}\int_1^b xe^{-x^2}dx=\lim_{b\to\infty}\frac{-1}{2}e^{-x^2}\Big|_1^b=\lim_{b\to\infty}\frac{-1}{2}e^{-b^2}+\frac{1}{2}e^{-1}=\frac{1}{2e}$. Since this integral converges, the series converges by the integral test.

4. Let $f(x)=\frac{1}{x\ln x}$. For $x\geq 2$, this function is positive, continuous, and decreasing.

$\int_2^{\infty}\frac{dx}{x\ln x}=\lim_{b\to\infty}\int_2^b\frac{dx}{x\ln x}=\lim_{b\to\infty}\ln(\ln x)\Big|_2^b=\lim_{b\to\infty}\ln(\ln b)-\ln(\ln 2)$. This limit diverges, albeit very slowly. Since the improper integral diverges, the series diverges by the integral test.

5. This series is a $p$-series with $p = 1/5 \leq 1$. The series diverges by the $p$-series test.
6. This series is a $p$-series with $p = e > 1$. The series converges by the $p$-series test.
7. $\frac{1}{n\sqrt{n}}=\frac{1}{n^{3/2}}$. This series is a $p$-series with $p = 3/2 > 1$. The series converges by the $p$-series test.
8. Let $f(x)=\frac{x^2}{\left(x^3+2\right)^4}$. For $x \geq 1$, this function is positive, continuous, and decreasing.

$\int_1^{\infty}\frac{x^2}{\left(x^3+2\right)^4}dx=\lim_{b\to\infty}\int_1^b\frac{x^2}{\left(x^3+2\right)^4}dx=\lim_{b\to\infty}\frac{-1}{9\left(x^3+2\right)^3}\Bigg|_1^b=\lim_{b\to\infty}\frac{-1}{9\left(b^3+2\right)^3}+\frac{1}{9\left(1^3+2\right)^3}=\frac{1}{243}$. Since this integral converges, the series converges by the integral test.

9. $\frac{3n^{1/3}}{2n^{2/5}}=\frac{3}{2}\cdot\frac{1}{n^{2/5-1/3}}=\frac{3}{2}\cdot\frac{1}{n^{1/15}}$. This is a $p$-series with $p = 1/15 \leq 1$. The series diverges by the $p$-series test.
10. Let $f(x)=\frac{1}{2x+5}$. For $x \geq 0$, this function is positive, continuous, and decreasing.

$\int_0^{\infty}\frac{dx}{2x+5}=\lim_{b\to\infty}\int_0^b\frac{dx}{2x+5}=\lim_{b\to\infty}\frac{1}{2}\ln(2x+5)\Big|_0^b=\lim_{b\to\infty}\frac{1}{2}\ln(2b+5)-\frac{1}{2}\ln 5$. This limit does not exist, so the integral diverges. Since the integral diverges, the series diverges by the integral test.

11. We will compare to $\sum_{n=1}^{\infty}\frac{1}{n^2}$ which is a convergent $p$-series ($p = 2 > 1$).

$\lim_{n\to\infty}\frac{\frac{1}{n^2}}{\frac{n^3+2n}{n^5+8}}=\lim_{n\to\infty}\frac{n^5+8}{n^2\left(n^3+2n\right)}=\lim_{n\to\infty}\frac{n^5+8}{n^5+2n^3}=1$ which is positive and finite. Therefore the given series converges by the limit comparison test.

12. We will compare to $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}$ a divergent $p$-series ($p = 1/2 \leq 1$). For all $n \geq 2$, $\frac{1}{\sqrt{n}-1}>\frac{1}{\sqrt{n}}$. Therefore the given series diverges by the direct comparison test.
13. We will compare to $\sum_{n=3}^{\infty}\frac{1}{\ln n}$ which diverges (see Example 4 – this is a useful series to compare against, so it is worth knowing that it diverges). For all $n \geq 3$, $\ln(\ln n)<\ln n$. It follows that $\frac{1}{\ln(\ln n)}>\frac{1}{\ln n}$. The given series diverges by the direct comparison test.

14. We will compare to $\sum_{n=0}^{\infty}\frac{3^n}{4^n}=\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n$ which is a convergent geometric series ($|r|=\frac{3}{4}<1$). For all $n$, $4^n+2>4^n$, so $\frac{1}{4^n+2}<\frac{1}{4^n}$, and finally $\frac{3^n}{4^n+2}<\frac{3^n}{4^n}$. By the direct comparison test, the given series converges.
15. We will compare to $\sum_{n=0}^{\infty}\frac{2^n}{n!}$ which converges by the ratio test. $\left(\lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}=\lim_{n\to\infty}\frac{2}{n+1}=0<1\right)$ For all $n$, $2^n-5<2^n$, so $\frac{2^n-5}{n!}<\frac{2^n}{n!}$. The given series converges by the direct comparison test.
16. We will compare to $\sum_{n=0}^{\infty}\frac{3^n}{4^n}=\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n$ which is still a convergent geometric series (see Problem 14). $\lim_{n\to\infty}\frac{\frac{3^n+2}{4^n}}{\frac{3^n}{4^n}}=\lim_{n\to\infty}\frac{\left(3^n+2\right)\cdot 4^n}{3^n\cdot 4^n}=1$. This limit is finite and positive, so the given series converges by the limit comparison test.
17. We will compare to $\sum_{n=2}^{\infty}\frac{1}{n}$ which is the harmonic series; it diverges. $\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{n^3-1}{n^4+1}}=\lim_{n\to\infty}\frac{n^4+1}{n\left(n^3-1\right)}=\lim_{n\to\infty}\frac{n^4+1}{n^4-n^3}=1$. This limit is positive and finite, so the given series diverges by the limit comparison test.
18. We will compare to $\sum_{n=0}^{\infty}\frac{1}{4^n}=\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n$ which is a convergent geometric series ($|r|=\frac{1}{4}<1$). $\lim_{n\to\infty}\left(\frac{1}{2n+4^n}\cdot\frac{4^n}{1}\right)=\lim_{n\to\infty}\frac{4^n}{4^n+2n}=1$. This limit is finite and positive, so the given series converges by the limit comparison test.
19. Note that depending on the value of $b$, the terms of this series may be negative for some values of $n$. However, since $a$ is positive, eventually $an$ will be larger than $b$ and $(an+b)$ will be positive. It is important to point this out because the comparison tests only apply to series that are, ultimately, positive-term series. Let $N$ be the threshold after which $an+b$ is always positive. We will compare to $\sum_{n=N}^{\infty}\frac{1}{n}$. $\lim_{n\to\infty}\left(\frac{1}{n}\div\frac{1}{an+b}\right)=\lim_{n\to\infty}\frac{1}{n}\cdot\frac{an+b}{1}=\lim_{n\to\infty}\frac{an+b}{n}=a$. This limit is positive and finite (because it is given that $a$ is positive). Therefore the given series diverges by the limit comparison test.
20. $\sum_{n=0}^{\infty}\frac{2^n}{3^{n+1}}=\sum_{n=0}^{\infty}\frac{1}{3}\cdot\left(\frac{2}{3}\right)^n$. This series is geometric with $|r|=\frac{2}{3}<1$. It converges by the geometric series test.
21. $\sum_{n=1}^{\infty}\frac{1}{2n}$. This series diverges by Problem 19 ($a=2$, $b=0$) or by limit comparison to the harmonic series.
22. $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{3^{n+1}\cdot(n+1)^3}{(n+1)!}\cdot\frac{n!}{3^n\cdot n^3}=\lim_{n\to\infty}\frac{(n+1)^3}{n^3}\cdot\frac{3^{n+1}}{3^n}\cdot\frac{n!}{(n+1)!}=\lim_{n\to\infty}\frac{(n+1)^3}{n^3}\cdot 3\cdot\frac{1}{n+1}=0<1$. This series converges by the ratio test.
23. This series is geometric with $|r|=\frac{2}{7}<1$. The series converges by the geometric series test.
24. $\lim_{n\to\infty}\frac{(n+1)^2+2^{n+1}}{(n+1)!}\cdot\frac{n!}{n^2+2^n}=\lim_{n\to\infty}\frac{(n+1)^2+2^{n+1}}{n^2+2^n}\cdot\frac{n!}{(n+1)!}=\lim_{n\to\infty}\left(2\cdot\frac{1}{n+1}\right)=0<1$. This series converges by the ratio test.
25. We will compare to $\sum\frac{1}{n}$, the divergent harmonic series. $\lim_{n\to\infty}\frac{\sin(1/n)}{1/n}=\lim_{n\to\infty}\frac{\left(-1/n^2\right)\cdot\cos(1/n)}{-1/n^2}$ by l'Hospital's rule applied to the indeterminate form 0/0. Continuing, $\lim_{n\to\infty}\frac{\left(-1/n^2\right)\cdot\cos(1/n)}{-1/n^2}=\lim_{n\to\infty}\cos\left(\frac{1}{n}\right)=\cos 0=1$. This limit is finite and positive. Therefore the given series diverges by the limit comparison test.
26. $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\cos\left(\frac{1}{n}\right)=\cos 0=1\neq 0$. This series diverges by the $n^{\text{th}}$ term test.

27. $\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\sin(1/n)}{1/n} = 1 \neq 0$. (For a justification of the limit computation, see Problem 25.) This series diverges by the $n^{\text{th}}$ term test.
28. $\lim_{n\to\infty} n\cos\left(\frac{1}{n}\right)$ does not exist; it has the form $\infty\cdot 1$, which is not indeterminate. This series diverges by the $n^{\text{th}}$ term test.
29. We will compare to $\sum\frac{1}{n^2}$ which is a convergent $p$-series ($p = 2 > 1$). $\lim_{n\to\infty}\frac{(1/n)\cdot\sin(1/n)}{1/n^2} = \lim_{n\to\infty}\frac{\sin(1/n)}{1/n} = 1$. (Again, for a justification of this limit, see Problem 25.) This limit is positive and finite, so the series converges.
30. We will compare to $\sum\frac{1}{n}$ which is the divergent harmonic series. $\lim_{n\to\infty}\frac{(1/n)\cdot\cos(1/n)}{1/n} = \lim_{n\to\infty}\cos\left(\frac{1}{n}\right) = 1$. This limit is positive and finite. Therefore the given series diverges by the limit comparison test.
31. $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{3^{n+1}\cdot(n+1)!}{(n+2)!}\cdot\frac{(n+1)!}{3^n\cdot n!} = \lim_{n\to\infty}\frac{3^{n+1}}{3^n}\cdot\frac{(n+1)!}{(n+2)!}\cdot\frac{(n+1)!}{n!} = \lim_{n\to\infty} 3\cdot\frac{1}{n+2}\cdot(n+1) = 3 > 1$. This series diverges by the ratio test. Though perhaps it would have been easier to simplify the factorials before starting…
32. $\sum_{n=2}^{\infty}\left(\frac{1}{n^3}-\frac{1}{n^4}\right) = \sum_{n=2}^{\infty}\frac{n-1}{n^4}$. Compare to the convergent $p$-series $\sum\frac{1}{n^3}$ ($p = 3 > 1$).
$\lim_{n\to\infty}\left(\frac{n-1}{n^4}\div\frac{1}{n^3}\right) = \lim_{n\to\infty}\frac{n^3(n-1)}{n^4} = \lim_{n\to\infty}\frac{n^4-1}{n^4} = 1$. This limit is positive and finite. Therefore the given series converges by the limit comparison test.
33. Let $f(x) = \frac{1}{x^2}\cdot e^{1/x}$. For $x \geq 1$, this is a positive, continuous, decreasing function.
$\int_1^{\infty}\frac{1}{x^2}e^{-1/x}dx = \lim_{b\to\infty}\int_1^b\frac{1}{x^2}e^{1/x}dx = \lim_{b\to\infty}\left(-e^{1/x}\right)\Big|_1^b = \lim_{b\to\infty}\left(-e^{1/b}+e^1\right) = e-1$. Because this integral converges, the corresponding series converges by the integral test.

There are, of course, many approaches to Problems 34-40. The candidates for geometric series, $p$-series, and ratio tests should be fairly obvious. Beyond those, students have a choice. The strategies below make up just one possible solution set.
34. This series diverges by the integral test. $f(x) = \frac{1}{2x+1}$ (which is positive, continuous, and decreasing) and $\int_0^{\infty}\frac{1}{2x+1}dx$ diverges.
35. This series diverges by the $n^{\text{th}}$ term test. $\lim_{n\to\infty} n = \infty \neq 0$.
36. This series diverges by the $p$-series test. $p = 1/2 \leq 1$.
37. This series converges by the geometric series test. $|r| = \frac{3}{4} < 1$.
38. This series diverges by the ratio test. $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)!}{3^{n+1}}\cdot\frac{3^n}{n!} = \lim_{n\to\infty}\frac{n+1}{3} = \infty > 1$.
39. This series converges by limit comparison to $\sum\frac{1}{n^2}$. $\lim_{n\to\infty}\frac{3}{n^2-1}\cdot\frac{n^2}{1} = 3$, which is positive and finite.
40. This series diverges by direct comparison to $\sum\frac{1}{n}$. $\frac{e^n}{n} > \frac{1}{n}$ for $n > 0$.

There are many approaches to Problems 41-49. See the comments before Problem 34.
41. This series diverges by the $p$-series test. $\sum\frac{n}{n^2} = \sum\frac{1}{n}$; $p = 1 \leq 1$.
42. This series diverges by limit comparison to $\sum\frac{1}{n}$. $\lim_{n\to\infty}\frac{n}{n^2-4}\cdot\frac{n}{1} = \lim_{n\to\infty}\frac{n^2}{n^2-4} = 1$ which is positive and finite.

43. This series diverges by the integral test. Let $f(x)=\frac{x}{x^2+4}$, which is positive, continuous, and decreasing for $x > 2$. $\int_3^\infty \frac{x}{x^2+4}dx$ diverges.

44. This series is a convergent telescoping series. $s_n = -2+\frac{2}{n+2}$. $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(-2+\frac{2}{n+2}\right) = -2$.

45. This series diverges by comparison to $\sum \frac{1}{n\ln n}$ which diverges (see Problem 4). For $n > 1$, $\sqrt{n} < n$. This implies that $\sqrt{n}\ln n < n\ln n$, and therefore $\frac{1}{\sqrt{n}\ln n} > \frac{1}{n\ln n}$.

46. This series converges by the geometric series test. $|r| = \frac{1}{4} < 1$.

47. This series converges by the ratio test. $\lim_{n\to\infty}\frac{(n+1)^2}{3^{n+1}}\cdot\frac{3^n}{n^2} = \lim_{n\to\infty}\frac{1}{3}\frac{(n+1)^2}{n^2} = \frac{1}{3} < 1$.

48. This series diverges by the $n^{\text{th}}$ term test. $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \frac{1}{e} \neq 0$.

49. This series converges by the root test. $\lim_{n\to\infty}\sqrt[n]{\left(\frac{3}{2n+1}\right)^n} = \lim_{n\to\infty}\frac{3}{2n+1} = 0 < 1$.

50. $\sum \frac{1}{\sqrt[4]{n}}$: This series is a divergent $p$-series ($p$ = 1/4 ≤ 1). If we apply the ratio test, we obtain the following limit: $\lim_{n\to\infty}\frac{1}{\sqrt[4]{n+1}}\cdot\frac{\sqrt[4]{n}}{1} = \lim_{n\to\infty}\frac{\sqrt[4]{n}}{\sqrt[4]{n+1}} = \lim_{n\to\infty}\sqrt[4]{\frac{n}{n+1}} = 1$.
$\sum \frac{1}{n^3}$: This series is a convergent $p$-series ($p$ = 3 > 1). If we apply the ratio test, we obtain the following limit: $\lim_{n\to\infty}\frac{1}{(n+1)^3}\cdot\frac{n^3}{1} = \lim_{n\to\infty}\frac{n^3}{(n+1)^3} = 1$.
The point here is that the ratio test really in inconclusive if the limit evaluates to 1. In this problem we see two examples where the limit is 1, and the respective series have different convergence behaviors.

51. We will apply the integral test to $\sum \frac{1}{n(\ln n)^p}$. Let $f(x)=\frac{1}{x(\ln x)^p}$. If $p$ is positive (as we have assumed), then this function is positive, continuous, and decreasing for $x \geq 2$. We need to examine the integral $\int_2^\infty \frac{1}{x(\ln x)^p}dx$. If we let $u = \ln(x)$ so that $du = \frac{1}{x}dx$, then we have $\int_2^\infty \frac{1}{x(\ln x)^p}dx = \int_{\ln 2}^\infty \frac{1}{u^p}du$. As we know, this integral converges iff $p > 1$. Therefore, the series converges iff $p > 1$ by the integral test.

52. We already know (Problem 4) that this series diverges if $p = 1$. We will consider separately the cases where $p > 1$ and $0 < p < 1$.
$p > 1$: For all $n \geq 3$, $\ln(n) > 1$. This implies that $n^p\ln(n) > n^p$, and finally $\frac{1}{n^p\ln(n)} < \frac{1}{n^p}$. If $p > 1$, then $\sum \frac{1}{n^p}$ converges by the $p$-series test. Therefore $\sum \frac{1}{n^p\ln(n)}$ converges by direct comparison.
$p < 1$: If $p < 1$, then $n > n^p$ for all $n > 1$. From this it follows that $n\ln n > n^p\ln n$, and finally $\frac{1}{n\ln n} < \frac{1}{n^p\ln n}$. As we know, $\sum \frac{1}{n\ln n}$ diverges. Therefore $\sum \frac{1}{n^p\ln n}$ diverges by direct comparison if $p < 1$.
Putting it all together, we see that $\sum \frac{1}{n^p\ln n}$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

53. For all $n \geq 3$, $\ln(n) > 1$. This implies that $\frac{\ln n}{n^p} > \frac{1}{n^p}$ for $n \geq 3$. If $p \leq 1$, then $\sum \frac{1}{n^p}$ diverges. Therefore, by direct comparison, $\sum \frac{\ln n}{n^p}$ diverges for $p \leq 1$.
It will take a little more work to show that the series converges when $p > 1$. We will use the integral test. Let $f(x)=\frac{\ln x}{x^p}$. If $p > 1$, this function is positive, continuous, and decreasing for $n \geq 2$. We need to integrate $\int_2^\infty \frac{\ln x}{x^p}dx$. We will do so by parts, letting $u = \ln(x)$ and $dv = x^{-p}dx$. This gives $du = \frac{1}{x}dx$ and $v = \frac{1}{-p+1}x^{-p+1} = \frac{1}{1-p}x^{1-p} = \frac{1}{1-p}\cdot\frac{1}{x^{p-1}}$.

$$\int_2^\infty \frac{\ln x}{x^p}dx = \lim_{b\to\infty}\left(\ln x\cdot\frac{1}{1-p}\cdot\frac{1}{x^{p-1}}\right)\Bigg|_2^b - \int_2^\infty \frac{1}{x}\cdot\frac{1}{1-p}\cdot\frac{1}{x^{p-1}}dx = \lim_{b\to\infty}\left(\frac{\ln x}{(1-p)x^{p-1}}\right)\Bigg|_2^b - \frac{1}{1-p}\int_2^\infty \frac{1}{x^p}dx$$

We know from our study of improper integrals that the integral at the end of this expression will converge if $p > 1$. All we really need to determine is whether the limit in the above expression is finite as well. Partially evaluating that limit gives the following:

$$\lim_{b\to\infty}\left(\frac{\ln x}{(1-p)x^{p-1}}\right)\Bigg|_2^b = \frac{1}{1-p}\cdot\lim_{b\to\infty}\frac{\ln x}{x^{p-1}} - \frac{1}{1-p}\cdot\frac{\ln 2}{2^{p-1}}.$$

Again, all that really matters is whether this is finite as $b\to\infty$. Indeed, since $p > 1$, $\lim\limits_{b\to\infty}\frac{\ln x}{x^{p-1}}$ is finite. If you don't believe me, begin by using l'Hospital's rule:

$$\lim_{b\to\infty}\frac{\ln x}{x^{p-1}} = \lim_{b\to\infty}\frac{1/x}{(p-1)x^{p-2}} = \lim_{b\to\infty}\frac{1}{(p-1)\cdot x^{p-1}} = 0.$$

Tracking backwards, we see that the original improper integral converges. Therefore, by the integral test, the series converges for $p > 1$.

54. To analyze the convergence of $\sum\limits_{n=1}^{\infty}\frac{1}{p_n}$, we will compare to $\sum\limits_{n=2}^{\infty}\frac{1}{n\ln n}$ which we know (from Problem 4) diverges. $\lim\limits_{n\to\infty}\left(\frac{1}{n\ln n}\div\frac{1}{p_n}\right) = \lim\limits_{n\to\infty}\frac{p_n}{n\ln n} = 1$ (by the Prime Number Theorem). This limit is positive and finite, so the series $\sum\frac{1}{p_n}$ diverges by the limit comparison test.
55. We apply the ratio test. $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\frac{1}{f_{n+1}}\cdot\frac{f_n}{1} = \lim\limits_{n\to\infty}\frac{f_n}{f_{n+1}} = \frac{1}{\phi}\approx 0.618 < 1$. The series $\sum\frac{1}{f_n}$ converges by the ratio test.
56. Let $d_1$ be the degree of $p_1(x)$ and let $d_2$ be the degree of $p_2(x)$. Then $\sum\limits_{n=N}^{\infty}\frac{p_1(n)}{p_2(n)}$ converges as long as $d_2 \ge d_1 + 2$. In general, one would use limit comparison against $\sum\frac{1}{n^d}$, where $d$ is the difference between $d_2$ and $d_1$. Such a comparison will always produce a finite, positive limit (either the ratio of leading coefficients of the polynomials or the reciprocal of that number, depending on how the limit is set up). $\sum\frac{1}{n^d}$ is a $p$-series; it converges only if $d$ is at least two. (Note that the difference between the degrees of two polynomials cannot be between 1 and 2 since polynomials have integer degrees.) Therefore we must have $d_2 - d_1 \ge 2$ for convergence of $\sum\limits_{n=N}^{\infty}\frac{p_1(n)}{p_2(n)}$.
57. Answers will vary. For example, if $a_n = 2^n$ and $b_n = 3^n$, then $\Sigma a_n$ and $\Sigma b_n$ both diverge (geometric with $|r| > 1$), but $\Sigma a_n/b_n$ converges ($|r| = 2/3 < 1$).
58. Answers will vary. For example $a_n = \frac{1}{2^n}$ and $b_n = \frac{1}{3^n}$ so that $\frac{a_n}{b_n} = \left(\frac{3}{2}\right)^n$.
59. Answers will vary. For example, $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$ so that $\frac{a_n}{b_n} = \frac{1/n^2}{1/n} = \frac{n}{n^2} = \frac{1}{n}$.
60. Answers will vary. For example, $a_n = \frac{1}{n}$ and $b_n = \frac{1}{\sqrt{n}}$ so that $\frac{a_n}{b_n} = \frac{1/n}{1/\sqrt{n}} = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$
61. It cannot be done. If $\Sigma b_n$ converges and $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = 0$, then $\Sigma a_n$ must converge as well. This is a strengthening of the limit comparison test.
62. For most combinations of convergence and divergence behavior, it is possible to come up with series $\Sigma a_n$ and $\Sigma b_n$ such that $\frac{a_n}{b_n}\to\infty$.

    Both $\Sigma a_n$ and $\Sigma b_n$ converge: $a_n = \frac{1}{n^2}$, $b_n = \frac{1}{n^3}$, $\frac{a_n}{b_n} = n \to \infty$

Both $\Sigma a_n$ and $\Sigma b_n$ diverge: $a_n = \frac{1}{\sqrt{n}}$, $b_n = \frac{1}{n}$, $\frac{a_n}{b_n} = \sqrt{n} \to \infty$

$\Sigma a_n$ diverges and $\Sigma b_n$ converges: $a_n = \frac{1}{n}$, $b_n = \frac{1}{n^2}$, $\frac{a_n}{b_n} = n \to \infty$

However, one cannot come up with an example in which $\Sigma a_n$ converges, $\Sigma b_n$ diverges, and $\frac{a_n}{b_n} \to \infty$. If $\frac{a_n}{b_n} \to \infty$ and $\Sigma b_n$ diverges, then $\Sigma a_n$ must diverge as well.

63. We will use limit comparison, comparing to the harmonic series. $\lim_{n\to\infty} \frac{a_n}{1/n} = \lim_{n\to\infty} n a_n = L$. Since this limit is finite and positive, $\Sigma a_n$ must diverge by the limit comparison test.

64. No. Suppose $a_n = \frac{1}{n}$. Then $\sum \frac{a_n}{n} = \sum \frac{1}{n^2}$ which converges. However, $\sum a_n = \sum \frac{1}{n}$ still diverges.

## Section 8

1. The series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$. We will skip over checking for convergence and go straight to absolute convergence. $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^3}$. This is a convergent $p$-series ($p = 3 > 1$), so the absolute value series converges. Therefore the given series converges absolutely.

2. The series is $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{(n+1)}{n^2}$. This series is alternating, with $a_n = \frac{(n+1)}{n^2}$. $\frac{n+2}{(n+1)^2} < \frac{n+1}{n^2}$ for $n \geq 1$, and $\lim_{n\to\infty} \frac{n+1}{n^2} = 0$. Therefore the series converges by the AST.

   We now turn to the absolute value series $\sum_{n=1}^{\infty} \left| (-1)^{n+1} \cdot \frac{(n+1)}{n^2} \right| = \sum_{n=1}^{\infty} \frac{(n+1)}{n^2}$. By limit comparison to the harmonic series, this series diverges. ($\lim_{n\to\infty} \frac{(n+1)/n^2}{1/n} = \lim_{n\to\infty} \frac{n(n+1)}{n^2} = 1$ which is positive and finite.) Therefore the given series converges conditionally.

3. Consider the absolute value series $\sum_{n=0}^{\infty} \left| 3 \cdot \left(\frac{-1}{2}\right)^n \right| = \sum_{n=0}^{\infty} 3 \cdot \left(\frac{1}{2}\right)^n$. This is a convergent geometric series ($|r| = \frac{1}{2} < 1$). Therefore the given series converges absolutely.

4. This is an alternating series with $a_n = \frac{\sqrt{n}}{n+1}$. $\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$ and $\frac{\sqrt{n+1}}{n+2} < \frac{\sqrt{n}}{n+1}$ for all $n \geq 1$. The series converges by the AST.

   The absolute value series is $\sum_{n=1}^{\infty} \left| (-1)^n \cdot \frac{\sqrt{n}}{n+1} \right| = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$. This series diverges by comparison to the divergent $p$-series $\sum \frac{1}{\sqrt{n}}$ ($p = 1/2 \leq 1$). ($\lim_{n\to\infty} \frac{1/\sqrt{n}}{\sqrt{n}/(n+1)} = \lim_{n\to\infty} \frac{n+1}{n} = 1$, which is positive and finite.) Therefore the given series is conditionally convergent.

5. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \cdot \frac{2n+1}{3n-4}$ does not exist and therefore is not zero. This series is divergent, as proven by the $n^{\text{th}}$ term test.

6. The absolute value series is $\sum_{n=0}^{\infty} \left| \frac{(-2)^n}{n!} \right| = \sum_{n=0}^{\infty} \frac{2^n}{n!}$. This series converges by the ratio test: $\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$. Therefore the given series converges absolutely.

7. This is a convergent $p$-series ($p = 4 > 1$). Since it is a positive-term series, its convergence is automatically absolute convergence.

8. This is a geometric series with $|r| = 1.2 > 1$. The series diverges by the geometric series test.
9. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}+1}{(n+1)!} \cdot \frac{n!}{3^n+1} = \lim_{n\to\infty} \frac{3^{n+1}+1}{3^n+1} \cdot \frac{n!}{(n+1)!} = 0 < 1$. This positive-term series converges by the ratio test. The convergence is absolute because the series is positive-term.
10. This series is alternating with $a_n = \frac{1}{2n+1}$. $\lim_{n\to\infty} \frac{1}{2n+1} = 0$ and $\frac{1}{2(n+1)+1} = \frac{1}{2n+3} < \frac{1}{2n+1}$ for all $n \ge 0$. The series converges by the AST.
The absolute value series is $\sum_{n=0}^{\infty} \left|\frac{\cos(n\pi)}{2n+1}\right| = \sum_{n=0}^{\infty} \frac{1}{2n+1}$. This series diverges (see Problem 19 of Section 7). Therefore the given series converges conditionally.
11. Because of the dominance of the factorial over the exponential, $\lim_{n\to\infty} \frac{(-1)^n \cdot n!}{10^n}$ does not exist. The series diverges.
12. This series is not strictly alternating. However, consider the absolute value series. $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n(n+1)/2}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$.
This is a convergent $p$-series ($p = 2 > 1$). Therefore the given series converges absolutely.
13. Because the cosine function is bounded by 1, $2 - \cos(n)$ is always positive. This is a positive term series. We will compare it to $\sum \frac{4}{n^2}$ which is a convergent $p$-series ($p = 2 > 1$). $\frac{2-\cos n}{n^2} < \frac{4}{n^2}$, so the given series converges by direct comparison. Since it is a positive-term series, the convergence is absolute.
14. This series does not strictly alternate, but consider the absolute value series: $\sum_{n=1}^{\infty} \left|\frac{\sin n}{3^n}\right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{3^n}$. This is a positive term series and can be compared to the convergent geometric series $\sum \frac{1}{3^n}$ ($|r| = \frac{1}{3} < 1$). Since $|\sin n| \le 1$, $\frac{|\sin n|}{3^n} \le \frac{1}{3^n}$. The absolute value series converges by direct comparison, and so the original series converges absolutely.
15. The absolute value series is $\sum_{n=0}^{\infty} \left|\frac{(-3)^n}{n!+n^2}\right| = \sum_{n=0}^{\infty} \frac{3^n}{n!+n^2}$. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!+(n+1)^2} \cdot \frac{n!+n^2}{3^n}$
$= \lim_{n\to\infty} \left(3 \cdot \frac{n!+n^2}{(n+1)!+(n+1)^2}\right) = 0 < 1$. The series converges by the ratio test. Therefore the original series converges absolutely.
16. $s_{10} = \sum_{n=1}^{10} \frac{(-1)^{n+1}}{n \cdot 2^n} = 0.4054346$. $|a_{11}| = 0.0000444$; this is maximum possible error in the approximation.
17. $s_{15} = \sum_{n=1}^{15} (-1)^n \cdot \frac{n}{n^3+1} = -0.3491$. The maximum possible error in this approximation is $|a_{16}| = 0.00391$.
18. $s_{20} = \sum_{n=1}^{20} \frac{n^2}{(-3)^{n+1}} = 0.03125$. The maximum possible error in this approximation is $|a_{21}| = 1.405 \times 10^{-8}$.
19. We require that $|a_{n+1}| < 0.05$. $\frac{1}{n+1} < 0.05 \Rightarrow 20 < n+1 \Rightarrow n > 19$. As long as we use at least 20 terms, we will have the desired accuracy.
20. We require that $|a_{n+1}| < 0.05$. $\frac{n+1}{(n+1)^3+10} < 0.05 \Rightarrow n \ge 4$. As long as we use at least 4 terms, we will have the desired accuracy.
21. We require that $|a_{n+1}| < 0.0005$. $\frac{1}{(n+1)^3} < 0.0005 \Rightarrow (n+1)^3 > 2000 \Rightarrow (n+1) > 12 \Rightarrow n > 11$. As long as we use at least 12 terms, we will have the desired accuracy.
22. We require that $|a_{n+1}| < 0.00005$. The $\cos(n\pi)$ factor accounts for the alternation of the series; we only need to consider the $1/(n^2 - 1)$. $\frac{1}{(n+1)^2-1} < 0.00005 \Rightarrow (n+1)^2 - 1 > 20000 \Rightarrow (n+1)^2 > 200001$

$\Rightarrow n+1>141$. So we must take $n$ to be at least 141. However, the question is about how many *terms* are needed, and the index variable begins with $n$ = 2, not $n$ = 1. Therefore, the answer to the question being asked is that we should use at least 140 terms.

23. $s_{10}=\sum_{n=1}^{10}\frac{(-1)^{n-1}}{\sqrt{n}}=0.450725$. $|a_{11}|=\frac{1}{\sqrt{11}}=0.3015$. Therefore bounds for the actual value of the series are given by $0.451-0.302\le\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}\le 0.451+0.302$ or $0.149\le\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}\le 0.752$.
24. $s_5=\sum_{n=0}^{5}\frac{\cos(n\pi)}{(2n)!}=0.540302303792$. $|a_6|=\frac{1}{12!}=2.088\times10^{-9}$. Therefore bounds for the actual value of the series are given by $0.540302303792-2.088\times10^{-9}\le\sum_{n=0}^{\infty}\frac{\cos(n\pi)}{(2n)!}\le 0.540302303792+2.088\times10^{-9}$ or $0.5403023017\le\sum_{n=0}^{\infty}\frac{\cos(n\pi)}{(2n)!}\le 0.5403023059$.
25. Absolutely convergent series "converge faster" because their terms are vanishing to zero much more quickly. Therefore the various decimal places of the partial sums stabilize earlier.
26. The actual value of an alternating series whose terms decrease monotonically in size must be between any two consecutive partial sums. Thus, if $s$ represents the value of the series, then we have $s_{100}<s<s_{101}$ and $s_{101}<s<s_{102}$. The latter bounds are tighter (as they must be since the error is always given by the next term), so we have $3.61<s<3.58$.
27. The actual value of an alternating series whose terms decrease monotonically in size must be between any two consecutive partial sums. Thus, if $s$ represents the value of the series, then we have $s_{40}<s<s_{41}$ and $s_{41}<s<s_{42}$. The latter bounds are tighter, so we have $11.956<s<12.089$.
28. $\cos(x)\approx 1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6$ so a fourth-order approximation of cos(-1) is given by $\cos(-1)\approx 1-\frac{1}{2}(-1)^2+\frac{1}{24}(-1)^4=1-\frac{1}{2}+\frac{1}{24}=\frac{13}{24}$. The maximum error is given by the first omitted term, in this case $\frac{1}{720}(-1)^6=\frac{1}{720}$.
29. $\sin x\approx x-\frac{1}{6}x^3+\frac{1}{120}x^5+\frac{1}{5040}x^7-\frac{1}{362880}x^9$. An approximation of sin(5) is therefore given by $\sin(5)\approx 5-\frac{1}{6}(5)^3+\frac{1}{120}(5)^5-\frac{1}{5040}(5)^7+\cdots+\frac{(-1)^n}{(2n+1)!}(5)^{2n+1}$. If we want error less than $10^{-6}$, then we need the first omitted term, $\frac{(-1)^{n+1}}{(2n+3)!}(5)^{2n+3}$ to be less than $10^{-6}$ in absolute value. Scanning a table of values, we see that $\frac{5^{2n+3}}{(2n+3)!}<10^{-6}$ when $n$ = 10. Thus, we need a polynomial of degree $2\cdot10+1=21$.

    For sin(0.5), we repeat the above analysis, but with 0.5 substituted for 5. $\frac{(0.5)^{2n+3}}{(2n+3)!}<10^{-6}$ when $n$ is 3. We need a polynomial of degree $2\cdot3+1=7$.

    Finally, when we scan a table to see when $\frac{(0.01)^{2n+3}}{(2n+3)!}<10^{-6}$, we have $n$ = 0. This means we need a polynomial of only degree $2\cdot0+1=1$. A first-degree (i.e., linear) polynomial is sufficient to approximate sin(0.01) with the desired accuracy.
30. a. $e^{-2}\approx P_5(-2)=1+(-2)+\frac{1}{2!}(-2)^2+\frac{1}{3!}(-2)^3+\frac{1}{4!}(-2)^4+\frac{1}{5!}(-2)^5$
    $=1-2+\frac{2^2}{2!}-\frac{2^3}{3!}+\frac{2^4}{4!}-\frac{2^5}{5!}=\frac{1}{15}$

    b. The error is no more than the absolute value of the first omitted term: $\frac{2^6}{6!}=\frac{4}{45}$

    c. The Lagrange error bound is given by $|R_5(-2)|\le\frac{M}{6!}\cdot|-2-0|^6$. The maximum value of the sixth derivative of $f(x)=e^x$ on the interval $[-2,0]$ is $e^0=1$, so we can take $M$ = 1. This gives an error bound of $\frac{1}{6!}(2)^6=\frac{4}{45}$.

d. While both methods give the same error bound in this case, the alternating series error bound is a little simpler to apply since we do not have to worry about an $M$-value.
e. We cannot repeat the comparison of errors for $x =$ 2. When we plug in 2 for $x$, we do not get an alternating series. In this case we can only use the Lagrange error bound.

31. a. To approximate ln(1.3), we let $x$ = 0.3 in the polynomial for $f(x)=\ln(1+x)$. This gives $P_5(0.3)=0.3-\frac{1}{2}(0.3)^2+\frac{1}{3}(0.3)^3-\frac{1}{4}(0.3)^4+\frac{1}{5}(0.3)^5=0.262461$.
b. The first omitted term from this approximation is $\frac{1}{6}(0.3)^6=0.0001215$. Therefore we have $0.262461-0.00012\le\ln(1.3)\le0.262461+0.00012$ or $0.26234\le\ln(1.3)\le0.26258$.
c. $\left|R_5(0.3)\right|\le\frac{M}{6!}(0.3-0)^6$. For $M$ we need a bound for the value of $f^{(6)}(x)$ on $[0,0.3]$. $\left|f^{(6)}(x)\right|=\left|\frac{-120}{(1+x)^6}\right|$ which takes on its maximum value in this interval at $x$ = 0. $\left|f^{(6)}(0)\right|=120$, so we use this value for $M$. $\left|R_5(0.3)\right|\le\frac{120}{6!}(0.3)^6=0.0001215$. The work involved in coming up with a value for $M$ makes the alternating series bound much more convenient.

32. a. $\cos(0.4)\approx1-\frac{0.4^2}{2!}+\frac{0.4^4}{4!}=0.92106666\ldots$.
b. The maximum error is the first omitted term: $\frac{0.4^6}{6!}=5.6889\times10^{-6}$
c. $\left|R_4(0.4)\right|\le\frac{M}{5!}\cdot(0.4)^5$. As usual for sine and cosine functions, we will take $M$ = 1. Then $\left|R_4(0.4)\right|\le\frac{0.4^5}{5!}=0.000085333\ldots$. The alternating series error bound is a little tighter and a little easier to generate.

33. a. $\arctan(1)\approx P_3(1)=1-\frac{1}{3}=\frac{2}{3}$. Hence $4\cdot\arctan(1)=4\cdot\frac{2}{3}=\frac{8}{3}=2.6666\ldots$. (I hope it goes without saying that this is a pretty wretched approximation for $\pi$.)
b. The maximum error in the approximation of arctan(1) is the first omitted term: 1/5. However, this means only that $\frac{2}{3}-\frac{1}{5}\le\arctan(1)\le\frac{2}{3}+\frac{1}{5}$. When we multiply by four, everything, including the error term, is scaled up by a factor of four. Therefore, the maximum possible error in our approximation for $\pi=4\cdot\arctan(1)$ is 4/5.
c. Accuracy to two decimal places normally means error less than 0.005. However, because of the factor of four, we actually have to have error less than $\frac{0.005}{4}=0.00125$. $\left|a_{n+1}\right|=\frac{1}{2(n+1)+1}=\frac{1}{2n+3}$. We need $\frac{1}{2n+3}<0.00125\Rightarrow 2n+3>800\Rightarrow 2n>797\Rightarrow n>398$. As long as we use an $n$-value of 399 or above, we will have the desired accuracy. Remember, though, that the initial term corresponds to $n$ = 0. So the number of terms corresponding to $n$ = 399 is 400. We need 400 terms to obtain the desired accuracy.

34. Answers will vary. One example: let $a_n=\frac{1}{n^2}$.
35. Answers will vary. One example: let $a_n=n$.
36. This is impossible. If the series is a positive-term series (as indicated), then any convergence is automatically absolute convergence. There is no such thing as a positive-term series that converges conditionally.
37. Answers will vary. One example: let $a_n=\frac{(-1)^n}{\sqrt{n}}$. $\sum a_n$ converges conditionally by the AST, but $\sum a_n^2=\sum\frac{1}{n}$ which diverges.
38. Answers will vary. One trivial example: let $a_n=\frac{1}{n}$ and $b_n=\frac{-1}{n}$.
39. False. Suppose $a_n=\frac{1}{n}$. Then $\sum a_n^2=\sum\frac{1}{n^2}$ converges as desired. However $\sum\left|\frac{1}{n}\right|$ diverges.
40. False. Suppose $a_n=\frac{(-1)^n}{\sqrt{n}}$. Then $-a_n=\frac{(-1)^{n+1}}{\sqrt{n}}$ and $\left|a_n\right|=\frac{1}{\sqrt{n}}$. $\sum a_n$ and $\sum -a_n$ both converge by the AST. However, $\sum\left|a_n\right|$ is a divergent $p$-series.

41. False. Let $a_n = \frac{1}{n}$. Then $\sum \frac{a_n}{n} = \sum \frac{1}{n^2}$ which converges. However, $\sum a_n = \sum \frac{1}{n}$ diverges.
42. True. This is Theorem 8.3.
43. False. This is true only for absolutely convergent series. A counter-example is $a_n = \frac{(-1)^n}{n}$.
44. True. One way to assess the convergence of $\sum (-1)^n a_n$ is to look at the corresponding absolute value series: $\sum \left|(-1)^n a_n\right| = \sum |a_n|$. We know this latter series converges because it is given that $\sum a_n$ converges absolutely. Therefore the series $\sum (-1)^n a_n$ must converge absolutely as well.
45. $\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim\limits_{n\to\infty} \left| \frac{(x+1)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(x+1)^n} \right| = \lim\limits_{n\to\infty} \left| \frac{n^2}{(n+1)^2} \cdot (x+1) \right| = |x+1|$. We require that $|x+1| < 1$.

$|x+1| < 1 \Rightarrow -1 < x+1 < 1 \Rightarrow -2 < x < 0$. We now check endpoints.

$x$ = -2: $\sum\limits_{n=1}^{\infty} \frac{(-2+1)^n}{n^2} = \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^2}$. This series converges (absolutely). The corresponding absolute value series is a convergent $p$-series ($p$ = 2 > 1).

$x$ = 0: $\sum\limits_{n=1}^{\infty} \frac{(0+1)^n}{n^2} = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$. This is a convergent $p$-series ($p$ = 2 > 1).

Radius of convergence: 1 Interval of convergence: $-2 \le x \le 0$
46. $\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim\limits_{n\to\infty} \left| \frac{\left((n+1)^2+3\right)x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{\left(n^2+3\right)x^n} \right| = \lim\limits_{n\to\infty} \left| \frac{(n+1)^2+3}{n^2+3} \cdot \frac{(2n)!}{(2n+2)!} \cdot x \right| = \lim\limits_{n\to\infty} \left| \frac{1}{(2n+2)(2n+1)} \cdot x \right| = 0$. This limit is less than 1, so the series converges for all real numbers.

Radius of convergence: $\infty$ Interval of convergence: $-\infty < x < \infty$
47. $\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim\limits_{n\to\infty} \left| \frac{(3x)^{n+1}}{2n+3} \cdot \frac{2n+1}{(3x)^n} \right| = \lim\limits_{n\to\infty} \left| \frac{2n+1}{2n+3} \cdot 3x \right| = |3x|$

$|3x| < 1 \Rightarrow -1 < 3x < 1 \Rightarrow \frac{-1}{3} < x < \frac{1}{3}$

$x = \frac{1}{3}$: $\sum\limits_{n=0}^{\infty} \frac{\left(3\cdot\frac{1}{3}\right)^n}{2n+1} = \sum\limits_{n=0}^{\infty} \frac{1}{2n+1}$. This series diverges. (See Section 7, Problem 19.)

$x = \frac{-1}{3}$: $\sum\limits_{n=0}^{\infty} \frac{\left(3\cdot\frac{-1}{3}\right)^n}{2n+1} = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. This series is alternating with $a_n = \frac{1}{2n+1}$. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{2n+1} = 0$ and $\frac{1}{2(n+1)+1} = \frac{1}{2n+3} < \frac{1}{2n+1}$ for all $n \ge 0$. Therefore the series converges by the AST.

Radius of convergence: 1/3 Interval of convergence: $-1/3 \le x < 1/3$
48. $\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim\limits_{n\to\infty} \left| \frac{(n+1)!\cdot(x-6)^{n+1}}{(n+1)^2+5} \cdot \frac{n^2+5}{n!\cdot(x-6)^n} \right| = \lim\limits_{n\to\infty} \left| \frac{n^2+5}{(n+1)^2+5} \cdot \frac{(n+1)!}{n!} \cdot (x-6) \right| = \lim\limits_{n\to\infty} \left| (n+1)\cdot(x-6) \right| = \infty$. This series diverges for all $x$ except $x$ = 6 (the center of the series).

Radius of convergence: 0 "Interval" of convergence: Converges only at $x$ = 6
49. This series is geometric with $r = \frac{x}{3}$. It converges iff $\left|\frac{x}{3}\right| < 1$. $\left|\frac{x}{3}\right| < 1 \Rightarrow \frac{|x|}{3} < 1 \Rightarrow |x| < 3 \Rightarrow -3 < x < 3$.

Radius of convergence: 3 Interval of convergence: $-3 < x < 3$.
50. $\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim\limits_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)\cdot 3^{n+1}} \cdot \frac{n\cdot 3^n}{x^n} \right| = \lim\limits_{n\to\infty} \left| \frac{n}{n+1} \cdot \frac{3^n}{3^{n+1}} \cdot x \right| = \frac{|x|}{3}$.

$\frac{|x|}{3} < 1 \Rightarrow |x| < 3 \Rightarrow -3 < x < 3$

$x$ = 3: $\sum\limits_{n=1}^{\infty} \frac{3^n}{n\cdot 3^n} = \sum\limits_{n=1}^{\infty} \frac{1}{n}$. This is the harmonic series. It diverges.

$x$ = -3: $\sum\limits_{n=1}^{\infty} \frac{(-3)^n}{n\cdot 3^n} = \sum\limits_{n=1}^{\infty} \frac{(-1)^n\cdot 3^n}{n\cdot 3^n} = \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$. This is the (opposite of the) alternating harmonic series. It converges.

Radius of convergence: 3 Interval of convergence: $-3 \le x < 3$.

51. $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2\cdot 3^{n+1}}\cdot\frac{n^2\cdot 3^n}{x^n}\right|=\lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\cdot\frac{3^n}{3^{n+1}}\cdot x\right|=\frac{|x|}{3}$

$\frac{|x|}{3}<1\Rightarrow|x|<3\Rightarrow -3<x<3$

$x$ = 3: $\sum_{n=1}^{\infty}\frac{3^n}{n^2\cdot 3^n}=\sum_{n=1}^{\infty}\frac{1}{n^2}$. This is a convergent $p$-series ($p$ = 2 > 1).

$x$ = -3: $\sum_{n=1}^{\infty}\frac{(-3)^n}{n^2\cdot 3^n}=\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 3^n}{n^2\cdot 3^n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$. The absolute-value version of this series is the one that was just considered. Since that series converges, this one converges absolutely.

Radius of convergence: 3 Interval of convergence: -3 ≤ $x$ ≤ 3.

52. $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\left|\frac{(n+1)!\cdot x^{n+1}}{3^{n+1}+(n+1)^2}\cdot\frac{3^n+n^2}{n!\cdot x^n}\right|=\lim_{n\to\infty}\left|\frac{3^n+n^2}{3^{n+1}+(n+1)^2}\cdot\frac{(n+1)!}{n!}\cdot x\right|=\lim_{n\to\infty}\left|\frac{1}{3}\cdot(n+1)\cdot x\right|=\infty$. This series converges only at its center.

Radius of convergence: 0 "Interval" of convergence: Converges only at $x = 0$

53. $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\left|\frac{(x-4)^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{(x-4)^n}\right|=\lim_{n\to\infty}\left|\frac{n^n}{(n+1)^{n+1}}\cdot(x-4)\right|=\lim_{n\to\infty}\left|\frac{n^n}{(n+1)\cdot(n+1)^n}\cdot(x-4)\right|=\lim_{n\to\infty}\left|\frac{1}{n+1}\cdot\left(\frac{n}{n+1}\right)^n\cdot(x-4)\right|$. The difficulty in evaluating this limit is in the middle factor. Observe that $\left(\frac{n}{n+1}\right)^n$ is the reciprocal of $\left(\frac{n+1}{n}\right)^n$. The latter expression turns out to be easier to evaluate. $\left(\frac{n+1}{n}\right)^n=\left(1+\frac{1}{n}\right)^n$ which tends to $e$ as $n\to\infty$. Therefore $\left(\frac{n}{n+1}\right)^n\to\frac{1}{e}$ as $n\to\infty$. Returning to the ratio test…

$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\left|\frac{1}{n+1}\cdot\left(\frac{n}{n+1}\right)^n\cdot(x-4)\right|=\lim_{n\to\infty}\frac{1}{n+1}\cdot\frac{|x-4|}{e}=0$. Since the limit is less than 1 independent of the value of $x$, this series converges for all $x$.

Radius of convergence: ∞ Interval of convergence: -∞ < $x$ < ∞

54. This series is geometric with $r=\frac{x+1}{5}$. It converges iff $\left|\frac{x+1}{5}\right|<1$.

$\left|\frac{x+1}{5}\right|<1\Rightarrow|x+1|<5\Rightarrow -5<x+1<5\Rightarrow -6<x<4$

Radius of convergence: 5 Interval of convergence: -6 < $x$ < 4

55. $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(n+1)!}\cdot\frac{n!}{x^{2n}}\right|=\lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\cdot\frac{x^{2n+2}}{x^{2n}}\right|=\lim_{n\to\infty}\left|\frac{x^2}{n+1}\right|=0$. Since the limit is less than 1 independent of the value of $x$, this series converges for all $x$.

Radius of convergence: ∞ Interval of convergence: -∞ < $x$ < ∞

56. $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\left|\frac{(x+2)^{2n+2}}{(n+1)^2+2(n+1)}\cdot\frac{n^2+2n}{(x+2)^{2n}}\right|=\lim_{n\to\infty}\left|\frac{n^2+2n}{(n+1)^2+2(n+1)}\cdot(x+2)^2\right|=(x+2)^2$

$(x+2)^2<1\Rightarrow|x+2|<1\Rightarrow -1<x+2<1\Rightarrow -3<x<-1$

$x$ = -1: $\sum_{n=1}^{\infty}\frac{(-1+2)^{2n}}{n^2+2n}=\sum_{n=1}^{\infty}\frac{1}{n^2+2n}$. This series converges by comparison to the convergent $p$-series $\sum\frac{1}{n^2}$ ($p=2>1$), as shown here: $\lim_{n\to\infty}\frac{1/n^2}{1/\left(n^2+2n\right)}=\lim_{n\to\infty}\frac{n^2+2n}{n^2}=1$, which is positive and finite.

$x$ = -3: $\sum_{n=1}^{\infty}\frac{(-3+2)^{2n}}{n^2+2n}=\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2+2n}=\sum_{n=1}^{\infty}\frac{1}{n^2+2n}$. This series still converges.

Radius of convergence: 1 Interval of convergence -3 ≤ $x$ ≤ -1

57. $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\left|\frac{\sqrt{n+1}\cdot(x+3)^{n+1}}{n+2}\cdot\frac{n+1}{\sqrt{n}\cdot(x+3)^n}\right|=\lim_{n\to\infty}\left|\frac{\sqrt{n+1}}{\sqrt{n}}\cdot\frac{n+1}{n+2}\cdot(x+3)\right|=|x+3|$

$|x+3|<1\Rightarrow -1<x+3<1\Rightarrow -4<x<-2$

$x$ = -2: $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}(-2+3)^n = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1} \cdot 1^n = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$. This series diverges by comparison to the divergent $p$-series $\sum \frac{1}{\sqrt{n}}$ ($p = 1/2 \le 1$): $\lim_{n\to\infty} \frac{1/\sqrt{n}}{\sqrt{n}/(n+1)} = \lim_{n\to\infty} \frac{n+1}{\sqrt{n}\cdot\sqrt{n}} = \lim_{n\to\infty} \frac{n+1}{n} = 1$, which is positive and finite.

$x$ = -4: $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}(-4+3)^n = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1} \cdot (-1)^n$. This series is alternating with $a_n = \frac{\sqrt{n}}{n+1}$. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$ and $\frac{\sqrt{n+1}}{(n+1+1)} < \frac{\sqrt{n}}{n+1}$ for all $n \ge 1$. Therefore this series converges by the AST.

Radius of convergence: 1 Interval of convergence: $-4 \le x < -2$

58. $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{3^{n+1} \cdot x^{n+1}}{2^{n+1}-1} \cdot \frac{2^n - 1}{3^n \cdot x^n} \right| = \lim_{n\to\infty} \left| \frac{3^{n+1}}{3^n} \cdot \frac{2^n-1}{2^{n+1}-1} \cdot x \right| = 3 \cdot \frac{1}{2} \cdot |x|$

$\frac{3}{2}|x| < 1 \Rightarrow -1 < \frac{3}{2} \cdot x < 1 \Rightarrow \frac{-2}{3} < x < \frac{2}{3}$

$x = \frac{2}{3}$: $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1} \cdot \left(\frac{2}{3}\right)^n = \sum_{n=1}^{\infty} \frac{6^n}{6^n - 3^n}$. $\lim_{n\to\infty} \frac{6^n}{6^n - 3^n} = 1 \ne 0$. This series diverges by the $n^{\text{th}}$ term test.

$x = \frac{-2}{3}$: $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1} \cdot \left(\frac{-2}{3}\right)^n$. This is the same series as the preceding one, except that it alternates.

Alternation will not help it pass the $n^{\text{th}}$ term test. This series diverges as well.

Radius of convergence: 2/3 Interval of convergence $\frac{-2}{3} < x < \frac{2}{3}$

59. We cannot use the ratio test in this example. The limit will not exist due to the oscillation of the sine function. This series is also not geometric. Our strategy will be the same as in Example 6 in which we bounded the value of the trigonometric function in order to use comparison.

$|\sin(n)| \le 1 \Rightarrow \frac{|\sin(n)|}{n^2} \le \frac{1}{n^2} \Rightarrow \frac{|\sin(n)|\cdot|x^n|}{n^2} \le \frac{|x^n|}{n^2}$. Therefore, if we can determine where $\sum \frac{|x^n|}{n^2}$ converges, it will follow that $\sum \frac{|\sin(n)|\cdot|x^n|}{n^2}$ will converge for the same $x$-values. It will then follow that the given series converges absolutely for those $x$-values.

$\sum_{n=1}^{\infty} \frac{|x^n|}{n^2}$: $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n} \right| = \lim_{n\to\infty} \left| \frac{n^2}{(n+1)^2} \cdot x \right| = |x|$. $|x| < 1 \Rightarrow -1 < x < 1$.

$x = \pm 1$: $\sum_{n=1}^{\infty} \frac{|(\pm 1)^n|}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$. Both endpoints generate the same convergent $p$-series ($p$ = 2 > 1).

Therefore this "helper series" converges for $-1 \le x \le 1$.

By the direct comparison test, it follows that $\sum_{n=1}^{\infty} \frac{|\sin(n)|\cdot|x^n|}{n^2}$ converges for $-1 \le x \le 1$. Finally, we conclude that the given series $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2} \cdot x^n$ converges absolutely on this interval.

Note that if $|x| > 1$, the factor $x^n$ will dominate the $n^2$ in the denominator. So we will have divergence outside this interval.

Radius of convergence: 1 Interval of convergence: $-1 \le x \le 1$

60. Answers will vary. For example: $\sum n!x^n$.

61. Answers will vary. For example: $\sum \frac{x^n}{n!}$.

62. Answers will vary, but geometric power series are good examples: $\sum x^n$.

63. Answers will vary. For example: $\sum \frac{x^n}{n^2}$.

64. Answers will vary. For example: $\sum_{n=1}^{\infty} \frac{-x^n}{n \cdot 2^n}$.

65. a. This series is geometric with $r = x$. It converges iff $|x| < 1$, i.e. on $-1 < x < 1$.

b. $f'(x) = \sum_{n=1}^{\infty} n \cdot x^{n-1} = \sum_{n=0}^{\infty} (n+1) \cdot x^n$. This result is obtained simply by differentiating the general term of the series. In the first summation, the index begins at $n = 1$ instead of $n = 0$ only because the general term of $f'(x)$ is 0 when $n = 0$; for this reason we have left it out of the series. The second summation is just a re-indexing of the first.

Differentiating a power series does not change its radius of convergence. The only possible change to the interval of convergence is the loss of endpoints, but the series for $f(x)$ did not converge at its endpoints to begin with.

Interval of convergence: $-1 < x < 1$.

c. $\int_0^x \left(1 + t + t^2 + t^3 + \cdots\right) dt = \left(t + \frac{1}{2}t^2 + \frac{1}{3}t^3 + \frac{1}{4}t^4 + \cdots\right)\Big|_0^x = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{1}{n}x^n + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n}$. Notice this is the same as what you'd obtain by simply integrating the general term of the series for *f*, give or take some re-indexing: $\int \sum_{n=0}^{\infty} x^n \, dx = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{1}{n} x^n$.

As in part (b), the radius of convergence will remain the same as for $f(x)$. We need only check the endpoints, since they may converge after integration.

$x = 1$: $\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$. This is the harmonic series. It diverges.

$x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. This is the (opposite of the) alternating harmonic series. It converges.

Interval of convergence: $-1 \le x < 1$.

66. a. $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{2n+2} \cdot \frac{2n}{(x-3)^n} \right| = \lim_{n\to\infty} \left| \frac{2n}{2n+2} \cdot (x-3) \right| = |x-3|$.

$|x-3| < 1 \Rightarrow -1 < x-3 < 1 \Rightarrow 2 < x < 4$

$x = 4$: $\sum_{n=1}^{\infty} \frac{(4-3)^n}{2n} = \sum_{n=1}^{\infty} \frac{1^n}{2n} = \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{1}{n}$. This is the harmonic series. It diverges.

$x = 2$: $\sum_{n=1}^{\infty} \frac{(2-3)^n}{2n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} = \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. This is the opposite of the alternating harmonic series. It converges.

Interval of convergence: $2 \le x < 4$.

b. $f'(x) = \sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{2} = \frac{1}{2} \cdot \sum_{n=1}^{\infty} (x-3)^{n-1} = \frac{1}{2} \cdot \sum_{n=0}^{\infty} (x-3)^n$. This series is geometric with $r = x - 3$. It converges iff $|x-3| < 1$. $|x-3| < 1 \Rightarrow -1 < x-3 < 1 \Rightarrow 2 < x < 4$.

Interval of convergence: $2 < x < 4$. (Note that we have "lost" the endpoint $x = 2$.)

c. $\int_3^x \left( \frac{t-3}{2} + \frac{(t-3)^2}{4} + \frac{(t-3)^3}{6} + \cdots \right) dt = \left( \frac{(t-3)^2}{2\cdot 2} + \frac{(t-3)^3}{3\cdot 4} + \frac{(t-3)^4}{4\cdot 6} + \cdots \right)\Big|_3^x = \frac{(x-3)^2}{2\cdot 2} + \frac{(x-3)^3}{3\cdot 4} + \frac{(x-3)^4}{4\cdot 6} + \cdots + \frac{(x-3)^n}{n\cdot 2(n-1)} + \cdots$ or $\sum_{n=2}^{\infty} \frac{(x-3)^n}{n\cdot 2(n-1)} = \sum_{n=1}^{\infty} \frac{(x-3)^{n+1}}{(n+1)\cdot 2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(x-3)^{n+1}}{n(n+1)}$, which is what we would have obtained from formally antidifferentiating the general term.

We need check only the endpoints of the interval from part (a).

$x = 4$: $\frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{(4-3)^n}{n(n+1)} = \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{1}{n^2+n}$. This series converges by comparison to the convergent $p$-series $\sum \frac{1}{n^2}$ $(p = 2 > 1)$: $\lim_{n\to\infty} \frac{1/n^2}{1/(n^2+n)} = \lim_{n\to\infty} \frac{n^2+n}{n^2} = 1$, which is positive and finite.

$x = 2$: $\frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{(2-3)^n}{n(n+1)} = \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+n}$. The absolute-value version of this series is the one just considered. This series converges absolutely.

Interval of convergence: $2 \le x \le 4$. (Note that we have "gained" the endpoint $x = 4$.)

67. a. $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \cdot (n+1)^2 \cdot (x+1)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(-1)^n \cdot n^2 \cdot (x+1)^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \cdot \frac{3^n}{3^{n+1}} \cdot (x+1) \right| = \frac{|x+1|}{3}$

$\frac{|x+1|}{3} < 1 \Rightarrow -1 < \frac{x+1}{3} < 1 \Rightarrow -3 < x+1 < 3 \Rightarrow -4 < x < 2$

$x = -4$: $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot n^2}{3^n} \cdot (-4+1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot n^2 \cdot (-3)^n}{3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot n^2 \cdot (-1)^n \cdot 3^n}{3^n} = \sum_{n=0}^{\infty} n^2$. This series clearly fails the $n^{\text{th}}$ term test. $\lim_{n\to\infty} n^2 = \infty$. The series diverges.

$x = 2$: $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot n^2}{3^n} (2+1)^n = \sum_{n=0}^{\infty} (-1)^n \cdot n^2$. This is like the previous series, but alternating. The alternation will not help the series pass the $n^{\text{th}}$ term test. This series diverges.

Interval of convergence: $-4 < x < 2$

b. $f(x) = 0 - \frac{1}{3}(x+1) + \frac{4}{9}(x+1)^2 - \frac{9}{27}(x+1)^3 + \cdots + \frac{(-1)^n \cdot n^2}{3^n}(x+1)^n + \cdots$

$f'(x) = \frac{-1}{3} + \frac{4}{9} \cdot 2(x+1) - \frac{9}{27} \cdot 3(x+1)^2 + \cdots + \frac{(-1)^n \cdot n^3}{3^n} \cdot (x+1)^{n-1} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^3}{3^n} \cdot (x+1)^{n-1}$

The radius of convergence for $f'(x)$ will be the same as for $f(x)$. Furthermore, since the series for $f(x)$ diverged at both endpoints, the series for $f'(x)$ will also diverge at both endpoints.

Interval of convergence: $-4 < x < 2$

c. We integrate term by term:

$$\int_{-1}^{x} \left( -\tfrac{1}{3}(t+1) + \tfrac{4}{9}(t+1)^2 - \tfrac{9}{27}(t+1)^3 + \cdots \right) dt = \left( \tfrac{-1}{3} \tfrac{(t+1)^2}{2} + \tfrac{4}{9} \tfrac{(t+1)^3}{3} - \tfrac{9}{27} \tfrac{(t+1)^4}{4} + \cdots \right) \Bigg|_{-1}^{x}$$

$$= \tfrac{-1}{3} \tfrac{(x+1)^2}{2} + \tfrac{4}{9} \tfrac{(x+1)^3}{3} - \tfrac{9}{27} \tfrac{(x+1)^4}{4} + \cdots + \tfrac{(-1)^n \cdot n^2}{3^n} \cdot \tfrac{(x+1)^{n+1}}{n+1} + \cdots$$

In summation notation: $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2}{3^n} \cdot \frac{(x+1)^{n+1}}{n+1}$.

The radius of convergence will be the same as that for $f(x)$. We need only check endpoints to see if we have "gained" either of them.

$x = -4$: $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2}{3^n} \cdot \frac{(-4+1)^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2}{3^n} \cdot \frac{(-3)^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2 \cdot (-1)^{n+1} \cdot 3^{n+1}}{3^n \cdot (n+1)} = -3 \cdot \sum_{n=1}^{\infty} \frac{n^2}{n+1}$. This series still fails the $n^{\text{th}}$ term test. $\lim_{n\to\infty} \frac{n^2}{n+1} = \infty \ne 0$. The series diverges.

$x = 2$: $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2}{3^n} \cdot \frac{(2+1)^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2}{3^n} \cdot \frac{3^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2 \cdot 3^{n+1}}{3^n \cdot (n+1)} = 3 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2}{n+1}$. Again, this is the same as the previous series, but alternating. It also fails the $n^{\text{th}}$ term test and diverges.

Interval of convergence: $-4 < x < 2$

68. a. $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{x^{n+1}}{2n+3} \cdot \frac{2n+1}{x^n} \right| = \lim_{n\to\infty} \left| \frac{2n+1}{2n+3} \cdot x \right| = |x|$

$|x| < 1 \Rightarrow -1 < x < 1$

$x=1$: $\sum_{n=0}^{\infty}\frac{1^n}{2n+1}=\sum_{n=0}^{\infty}\frac{1}{2n+1}$ This series diverges. (See Section 7, Problem 19.)

$x=-1$: $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$. This is an alternating series with $a_n=\frac{1}{2n+1}$. $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{2n+1}=0$. Also $\frac{1}{2(n+1)+1}=\frac{1}{2n+3}<\frac{1}{2n+1}$ for all $n\ge 0$. Therefore, this series converges by the AST.

Interval of convergence: $-1\le x<1$

b. $f(x)\approx 1+\frac{x}{3}+\frac{x^2}{5}+\frac{x^3}{7}+\frac{x^4}{9}$. $f(-1)\approx 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}=0.83492$.

c. Since the series for $f(-1)$ is alternating, the maximum possible error is the first omitted term, in this case 1/11. Therefore, we have $0.8349-0.0909\le f(-1)\le 0.8349+0.0909$ or $0.7440\le f(-1)\le 0.9258$.

69. a. $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\cdot(x-3)^{n+1}}{2^{n+1}\cdot(n+1)!}\cdot\frac{2^n\cdot n!}{(-1)^n\cdot(x-3)^n}\right|=\lim_{n\to\infty}\left|\frac{2^n}{2^{n+1}}\cdot\frac{n!}{(n+1)!}\cdot(x-3)\right|=\lim_{n\to\infty}\left|\frac{1}{2}\cdot\frac{1}{n+1}\cdot(x-3)\right|=0$. Since this limit is less than 1 for all *x*, the series converges for all *x*. The radius of convergence is infinite.

b. $f(x)\approx 1-\frac{(x-3)}{2\cdot 1!}+\frac{(x-3)^2}{2^2\cdot 2!}-\frac{(x-3)^3}{2^3\cdot 3!}=1-\frac{1}{2}(x-3)+\frac{1}{8}(x-3)^2-\frac{1}{48}(x-3)^3$

$f(4)\approx 1-\frac{1}{2}+\frac{1}{8}-\frac{1}{48}=\frac{29}{48}=0.60416666\ldots$.

c. Since the series for $f(4)$ is alternating, the error in a partial sum is given by the first omitted term. In this case, the error is no more than $\frac{1}{2^4\cdot 4!}=\frac{1}{384}=0.00260416666\ldots$.

d. In the series for $f(4)$, $|a_{n+1}|=\frac{1}{2^{n+1}\cdot(n+1)!}$. Solving $\frac{1}{2^{n+1}\cdot(n+1)!}<10^{-6}$ by table, we see that *n* must be at least 7. We need to use at least $P_7(4)$ to achieve the desired accuracy. Of course, there's a constant term in $P_7(x)$, so there are actually <u>eight</u> terms needed to guarantee error less than $10^{-6}$.

70. a. $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right|=\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}(x-a)\right|=\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|\cdot|x-a|=L|x-a|$. As usual, we require this limit to be less than 1. $L|x-a|<1\Rightarrow|x-a|<\frac{1}{L}$.

The radius of convergence of this series is $1/L$.

b. $|x-a|<\frac{1}{L}\Rightarrow-\frac{1}{L}<x-a<\frac{1}{L}\Rightarrow a-\frac{1}{L}<x<a+\frac{1}{L}$. The interval of convergence is $a-\frac{1}{L}<x<a+\frac{1}{L}$, give or take convergence at the endpoints.

# Section 10

1. $\cos(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{(-1)^n x^{2n}}\right|=\lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+1)}\right|=0<1$ for all *x*.

Interval of convergence: $-\infty<x<\infty$

2. By Taylor's Theorem, $\cos x=1-\frac{1}{2!}x^2+\cdots+\frac{(-1)^n}{(2n)!}x^{2n}+R_n(x)$ where $|R_n(x)|\le\frac{M}{(n+1)!}\cdot|x-0|^{n+1}$. Since the cosine function and all its derivatives are bounded by 1 for all *x*, we take *M* = 1. Then $|R_n(x)|=\frac{|x|^{n+1}}{(n+1)!}$. Taking the limit of the error bound, we have $\lim_{n\to\infty}|R_n(x)|\le\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!}=0$ since the factorial denominator will ultimately dominate the exponential numerator for any fixed *x*. This shows that in the limiting case of having infinitely many terms (i.e., the power series) the error goes to zero; the power series for the cosine function converges exactly to the function.

3. $\frac{1}{1-x} = 1 + x + x^2 + \cdots x^n + \cdots = \sum_{n=0}^{\infty} x^n$

This is a geometric series with common ratio $x$. It converges iff $|x| < 1$, i.e. on $-1 < x < 1$.
Interval of convergence: $-1 < x < 1$

4. $\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

This is a geometric series with common ratio $x^2$. It converges iff $x^2 < 1$. $x^2 < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$.
Interval of convergence: $-1 < x < 1$

5. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

This series is obtained by term-by-term integration of the series in Problem 4. Therefore, all we need to do is check for convergence at the endpoints

$x = 1$: $\sum_{n=0}^{\infty} \frac{(-1)^n (1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. This is an alternating series with $a_n = \frac{1}{2n+1}$. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2n+1} = 0$ and $\frac{1}{2(n+1)+1} = \frac{1}{2n+3} < \frac{1}{2n+1}$ for all $n \geq 0$. Therefore this series converges by the AST.

$x = -1$: $\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (-1)}{2n+1} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. This is the opposite of the series as for $x = 1$. It also converges.
Interval of convergence: $-1 \leq x \leq 1$

6. We begin by finding derivatives, dividing by factorials, and hoping to see a pattern.

$$\begin{array}{ccccc}
f(x) = \ln x & \to & f(1) = 0 & \to & 0/0! = 0 \\
f'(x) = x^{-1} & \to & f'(1) = 1 & \to & 1/1! = 1 \\
f''(x) = -x^{-2} & \to & f''(1) = -1 & \to & -1/2! = \frac{-1}{2} \\
f'''(x) = 2x^{-3} & \to & f'''(1) = 2 & \to & 2/3! = \frac{1}{3} \\
f^{(4)}(x) = -6x^{-4} & \to & f^{(4)}(1) = -6 & \to & -6/4! = \frac{-1}{4}
\end{array}$$

Generalizing, $\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + (-1)^{n+1} \cdot \frac{(x-1)^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$.

$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{(-1)^{n+2}(x-1)^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n+1}(x-1)^n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1}(x-1) \right| = |x-1|$.

$|x-1| < 1 \Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2$

$x = 0$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{-1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$. This is the (opposite of the) harmonic series; it diverges.

$x = 2$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. This is the alternating harmonic series; it converges.

Interval of convergence: $0 < x \leq 2$

Alternate approach: Since $\ln(x) = \ln(1 + (x-1)) = \ln(1 - -(x-1))$, we can simply substitute $-(x-1)$ in for $x$ in the series for $\ln(1-x)$ to obtain the same result. The interval of convergence shifts one unit to the right to give (0, 2].

7. To the tableau!

$$\begin{array}{ccccc}
f(x)=\cos x & \to & f\left(\frac{4\pi}{3}\right)=-1/2 & \to & \frac{-1/2}{0!}=\frac{-1}{2} \\
f'(x)=-\sin x & \to & f'\left(\frac{4\pi}{3}\right)=-\sqrt{3}/2 & \to & \frac{-\sqrt{3}/2}{1!}=\frac{-\sqrt{3}}{2} \\
f''(x)=-\cos x & \to & f''\left(\frac{4\pi}{3}\right)=1/2 & \to & \frac{1/2}{2!}=\frac{1}{4} \\
f'''(x)=\sin x & \to & f'''\left(\frac{4\pi}{3}\right)=\sqrt{3}/2 & \to & \frac{\sqrt{3}/2}{3!}=\frac{\sqrt{3}}{12} \\
f^{(4)}(x)=\cos(x) & \to & f^{(4)}\left(\frac{4\pi}{3}\right)=-1/2 & \to & \frac{-1/2}{4!}=\frac{-1}{48}
\end{array}$$

Therefore $\cos(x)=\frac{-1}{2}-\frac{\sqrt{3}}{2}\left(x+\frac{4\pi}{3}\right)+\frac{1}{4}\left(x+\frac{4\pi}{3}\right)^2+\frac{\sqrt{3}}{12}\left(x+\frac{4\pi}{3}\right)^3+\cdots$.

8. Again, we make a tableau.

$$\begin{array}{ccccc}
f(x)=e^x & \to & f(e)=e^e & \to & \frac{e^e}{0!}=e^e \\
f'(x)=e^x & \to & f'(e)=e^e & \to & \frac{e^e}{1!}=e^e \\
f''(x)=e^x & \to & f''(e)=e^e & \to & \frac{e^e}{2!} \\
f'''(x)=e^x & \to & f'''(e)=e^e & \to & \frac{e^e}{3!} \\
f^{(4)}(x)=e^x & \to & f^{(4)}(e)=e^e & \to & \frac{e^e}{4!}
\end{array}$$

Therefore $e^x=e^e+e^e(x-e)+\frac{e^e}{2!}(x-e)^2+\cdots+\frac{e^e}{n!}(x-e)^n+\cdots=\sum_{n=0}^{\infty}\frac{e^e(x-e)^n}{n!}$.

9. As we know, $\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$. We can just sub in $x-2$ for $x$ to get a series for $\sin(x-2)$.

$$\sin(x-2)=(x-2)-\frac{(x-2)^3}{3!}+\frac{(x-2)^5}{5!}-\cdots+(-1)^n\cdot\frac{(x-2)^{2n+1}}{(2n+1)!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^{2n+1}}{(2n+1)!}$$

10. Since $\arctan(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots+(-1)^n\frac{x^{2n+1}}{2n+1}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$, by substitution we have

$$\arctan\left(x^2\right)=x^2-\frac{\left(x^2\right)^3}{3}+\frac{\left(x^2\right)^5}{5}-\cdots+(-1)^n\frac{\left(x^2\right)^{2n+1}}{2n+1}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{2n+1}.$$

$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{4n+6}}{2n+3}\cdot\frac{2n+1}{(-1)^n x^{4n+2}}\right|=\lim_{n\to\infty}\left|\frac{2n+1}{2n+3}\cdot x^4\right|=x^4$. $x^4<1\Rightarrow|x|<1\Rightarrow-1<x<1$.

$x=1$: $\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 1^{4n+2}}{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$. This series converges; see Problem 5.

$x=-1$: $\sum_{n=1}^{\infty}\frac{(-1)^n\cdot(-1)^{4n+2}}{2n+1}=\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}$. This series still converges.

Interval of convergence: $-1\le x\le 1$

11. Since $e^x=1+x+\frac{x^2}{2!}+\cdots=\sum_{n=0}^{\infty}\frac{x^n}{n!}$, multiplying through by $x$ gives

$$xe^x=x\left(1+x+\frac{x^2}{2!}+\cdots\right)=x+x^2+\frac{x^3}{2!}+\cdots+\frac{x^{n+1}}{n!}+\cdots=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}$$

12. a. We start with a tableau.

$$
\begin{array}{lllll}
f(x) = 2xe^{x^2} & \to & f(0) = 0 & \to & 0 \\
f'(x) = \left(4x^2 + 2\right)e^{x^2} & \to & f'(0) = 2 & \to & \frac{2}{1!} \\
f''(x) = \left(8x^3 + 12x\right)e^{x^2} & \to & f''(0) = 0 & \to & 0 \\
f'''(x) = \left(16x^4 + 48x^2 + 12\right)e^{x^2} & \to & f'''(0) = 12 & \to & \frac{12}{3!} \\
f^{(4)}(x) = \left(32x^5 + 160x^3 + 120x\right)e^{x^2} & \to & f^{(4)}(0) = 0 & \to & 0 \\
f^{(5)}(x) = \left(64x^6 + 480x^4 + 720x^2 + 120\right)e^{x^2} & \to & f^{(5)}(0) = 120 & \to & \frac{120}{5!}
\end{array}
$$

Therefore we have the following for the Maclaurin series:

$$
\begin{aligned}
2xe^{x^2} &= 2x + \tfrac{12}{3!}x^3 + \tfrac{120}{5!}x^5 + \cdots = 2x + \tfrac{6\cdot 2}{3\cdot 2\cdot 1!}x^3 + \tfrac{60\cdot 2}{5\cdot 4\cdot 3\cdot 2!}x^5 + \cdots \\
&= 2x + \tfrac{2}{1!}x^3 + \tfrac{2}{2!}x^5 + \cdots + \tfrac{2}{n!}x^{2n+1} + \cdots = \sum_{n=0}^{\infty} \tfrac{2x^{2n+1}}{n!}
\end{aligned}
$$

(The seemingly arbitrary cancelling will become clear as we proceed through parts (b) and (c).)

b. Since $e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, it follows by substitution that

$e^{x^2} = 1 + x^2 + \frac{\left(x^2\right)^2}{2!} + \frac{\left(x^2\right)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$. We now multiply through by $2x$ to obtain

$$
\begin{aligned}
2xe^{x^2} &= 2x + 2x\cdot x^2 + 2x\cdot \tfrac{x^4}{2!} + 2x\cdot \tfrac{x^6}{3!} + \cdots \\
&= 2x + 2x^3 + \tfrac{2x^5}{2!} + \tfrac{2x^7}{3!} + \cdots + \tfrac{2x^{2n+1}}{n!} + \cdots = \sum_{n=0}^{\infty} \tfrac{2x^{2n+1}}{n!}.
\end{aligned}
$$

c. For this approach, we simply differentiate the series for $e^{x^2}$ term by term.

$$g(x) = e^{x^2} = 1 + x^2 + \tfrac{x^4}{2!} + \tfrac{x^6}{3!} + \cdots = \sum_{n=0}^{\infty} \tfrac{x^{2n}}{n!}$$

$$g'(x) = 2xe^{x^2} = 0 + 2x + \tfrac{4x^3}{2!} + \tfrac{6x^5}{3!} + \cdots = \sum_{n=1}^{\infty} \tfrac{2n\cdot x^{2n-1}}{n!} = \sum_{n=1}^{\infty} \tfrac{2x^{2n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \tfrac{2x^{2n+1}}{n!}$$

d. Yes!

13. After using the hint, we will substitute $2x$ into the series for $\cos(x)$.

$$
\begin{aligned}
\cos^2 x &= \tfrac{1}{2} + \tfrac{1}{2}\cos(2x) = \tfrac{1}{2} + \tfrac{1}{2}\cdot \sum_{n=0}^{\infty} \tfrac{(-1)^n\cdot(2x)^{2n}}{(2n)!} \\
&= \tfrac{1}{2} + \tfrac{1}{2}\left(1 - \tfrac{1}{2}\cdot 4\cdot x^2 + \tfrac{1}{24}\cdot 16\cdot x^4 - \cdots\right) = \tfrac{1}{2} + \tfrac{1}{2}\left(1 - 2x^2 + \tfrac{2}{3}x^4 - \cdots\right) = 1 - x^2 + \tfrac{1}{3}x^4 - \cdots
\end{aligned}
$$

14. Rather than attempt to multiply the series for sine and cosine, we will use a trig identity.

$$
\begin{aligned}
f(x) &= 2x^3 \sin x \cos x = x^3 \cdot 2\sin x \cos x = x^3 \sin(2x) \\
&= x^3 \cdot \sum_{n=0}^{\infty} \tfrac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \tfrac{(-1)^n\cdot 2^{2n+1}\cdot x^{2n+4}}{(2n+1)!} \\
&= 2x^4 - \tfrac{8}{3!}x^6 + \tfrac{32}{5!}x^8 - \cdots
\end{aligned}
$$

15. We begin with a tableau.

$$\begin{array}{cccccc} f(x)=\sqrt[3]{x}=x^{1/3} & \to & f(8)=2 & \to & \frac{2}{0!}=2 \\ f'(x)=\frac{1}{3}x^{-2/3} & \to & f'(8)=\frac{1}{12} & \to & \frac{1/12}{1!}=\frac{1}{12} \\ f''(x)=\frac{-2}{9}x^{-5/3} & \to & f''(8)=\frac{-1}{144} & \to & \frac{-1/144}{2!}=\frac{-1}{288} \\ f'''(x)=\frac{10}{27}x^{-8/3} & \to & f'''(8)=\frac{5}{3456} & \to & \frac{5/3456}{3!}=\frac{5}{20736} \end{array}$$

$f(x)=2+\frac{1}{12}(x-8)-\frac{1}{288}(x-8)^2+\frac{5}{20736}(x-8)^3-\cdots$

The radius of convergence of the series is 8. This is the distance from the center ($x$ = 8) to the $x$-coordinate of the vertical tangent ($x$ = 0). The function is not differentiable where its graph has a vertical tangent.

16. Notice that $\ln(4+x)=\ln(1+3+x)=\ln\left(1+(x+3)\right)$. We can simply substitute $x$ + 3 in for $x$ n the series for $g(x)=\ln(1+x)$.

$$\ln(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^2-\cdots+(-1)^{n+1}\cdot\tfrac{x^n}{n}+\cdots$$

$$\ln\left(1+(x+3)\right)=(x+3)-\tfrac{1}{2}(x+3)^2+\tfrac{1}{3}(x+3)^2-\cdots+(-1)^{n+1}\cdot\tfrac{(x+3)^n}{n}+\cdots=\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\tfrac{(x+3)^n}{n}$$

17. a. As usual, we make a tableau.

$$\begin{array}{cccccc} f(x)=(1+x)^k & \to & f(0)=1 & \to & \frac{1}{0!}=1 \\ f'(x)=k(1+x)^{k-1} & \to & f'(0)=k & \to & \frac{k}{1!}=k \\ f''(x)=k(k-1)(1+x)^{k-2} & \to & f''(0)=k(k-1) & \to & \frac{k(k-1)}{2!} \\ f'''(x)=k(k-1)(k-2)(1+x)^{k-3} & \to & f'''(0)=k(k-1)(k-2) & \to & \frac{k(k-1)(k-2)}{3!} \end{array}$$

From this we see that $f(x)=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\cdots+\frac{k(k-1)\cdot\cdots\cdot(k-(n-1))}{n!}x^n+\cdots$. Using combinations this can be rewritten as $f(x)=\binom{k}{0}+\binom{k}{1}x+\binom{k}{2}x^2+\cdots=\sum_{n=0}^{\infty}\binom{k}{n}x^n$ (though some authors only define this notation if $k$ is an integer).

b. The numerator of $\binom{k}{n}$ is $k(k-1)(k-2)\cdots(k-n+1)$. If $n$ is greater than $k$, then one of these factors will take the form ($k$ – $k$); it will vanish. Hence $\binom{k}{n}=0$ when $n > k$. If it is indeed the case that $n > k$, then only the terms up through degree $n$ will be non-zero. This means the series will terminate to generate a regular polynomial. In this case, we have $f(x)=\sum_{n=0}^{k}\binom{k}{n}x^n$.

If $k$ is an integer, the binomial theorem would give $\sum_{n=0}^{\infty}\binom{k}{n}x^n\cdot 1^{k-n}$, but since all powers of 1 are 1, this reduces to what we already have for $f(x)$.

c. We have $a_n=\frac{k(k-1)(k-2)\cdot\cdots\cdot(k-(n-2))(k-(n-1))}{n!}x^n$ and $a_{n+1}=\frac{k(k-1)\cdot\cdots\cdot(k-(n-2))(k-(n-1))(k-n)}{(n+1)!}x^{n+1}$. Now we are ready for the ratio test.

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\left|\frac{k(k-1)\cdot\cdots\cdot(k-n+1)(k-n)\cdot x^{n+1}}{(n+1)!}\cdot\frac{n!}{k(k-1)\cdot\cdots\cdot(k-n+1)\cdot x^n}\right|=\lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\cdot(k-n)\cdot x\right|=\lim_{n\to\infty}\left|\frac{k-n}{n+1}\cdot x\right|=|x|$$

We require that $|x|<1$, so the radius of convergence is 1.

d. i. Here we have $k$ = 5/2. $g(x)=1+\frac{5}{2}x+\frac{(5/2)(3/2)}{2!}x^2+\cdots+\frac{(5/2)(3/2)\cdot\cdots\cdot(5/2-n+1)}{n!}x^n+\cdots$

ii. $h(x)=2\left(1+x^2\right)^{-3}$. If we let $f(x)=(1+x)^{-3}$, then $h(x)=2f\left(x^2\right)$.

$$f(x)=1-3x+\tfrac{(-3)(-4)}{2!}x^2+\tfrac{(-3)(-4)(-5)}{3!}x^3+\cdots$$
$$f\left(x^2\right)=1-3x^2+\tfrac{(-3)(-4)}{2!}x^4+\tfrac{(-3)(-4)(-5)}{3!}x^6+\cdots$$
$$2f\left(x^2\right)=2\left(1-3x^2+\tfrac{(-3)(-4)}{2!}x^4+\tfrac{(-3)(-4)(-5)}{3!}x^6+\cdots\right)$$
$$h(x)=2\cdot\sum_{n=0}^{\infty}\tfrac{(-3)(-3-1)\cdot\cdots\cdot(-3-n+1)}{n!}x^{2n}$$

iii. $k(x)=\frac{1}{\sqrt{1-x^2}}=\left(1-x^2\right)^{-1/2}=\left(1+\left(-x^2\right)\right)^{-1/2}$. Let $f(x)=(1+x)^{-1/2}$ so that $k(x)=f\left(-x^2\right)$.

$$f(x)=1-\tfrac{1}{2}x+\tfrac{(-1/2)(-3/2)}{2!}x^2+\tfrac{(-1/2)(-3/2)(-5/2)}{3!}x^3+\cdots$$
$$f\left(-x^2\right)=1-\tfrac{1}{2}\left(-x^2\right)+\tfrac{(-1/2)(-3/2)}{2!}\left(-x^2\right)^2+\tfrac{(-1/2)(-3/2)(-5/2)}{3!}\left(-x^2\right)^3+\cdots$$
$$=1-\left(\tfrac{-1}{2}\right)x^2+\tfrac{(-1/2)(-3/2)}{2!}x^4-\tfrac{(-1/2)(-3/2)(-5/2)}{3!}x^6+\cdots$$
$$k(x)=\sum_{n=0}^{\infty}\tfrac{(-1/2)(-3/2)\cdot\cdots\cdot(-1/2-n+1)}{n!}x^{2n}$$

e. $\frac{d}{dx}\arccos(x)=-\frac{1}{\sqrt{1-x^2}}$. We simply need to integrate the series for $k(x)$, take its opposite, and make sure that the function obtained passes through the point (0, $\pi/2$).

$$\int k(x)dx=\int\left(1+\tfrac{1}{2}x^2+\tfrac{3}{8}x^4+\tfrac{5}{16}x^6+\cdots\right)dx=C+x+\tfrac{1}{6}x^3+\tfrac{3}{40}x^5+\tfrac{5}{112}x^7+\cdots$$
$$\arccos(x)=-\left(C+x+\tfrac{1}{6}x^3+\tfrac{3}{40}x^5+\tfrac{5}{112}x^7+\cdots\right)$$
$$=C-x-\tfrac{1}{6}x^3-\tfrac{3}{40}x^5-\tfrac{5}{112}x^7-\cdots$$

All terms in this series vanish except for the constant term when $x = 0$. Therefore we take $C=\pi/2$ to ensure agreement with the arccosine function.

$$l(x)=\tfrac{\pi}{2}-x-\tfrac{1}{6}x^3-\tfrac{3}{40}x^5-\tfrac{5}{112}x^7-\cdots$$

18. Here we have $k = 1/m$, so $(x+1)^{1/m}=1+\frac{1}{m}x+\frac{(1/m)(1/m-1)}{2!}x^2+\cdots$. Simplifying the quadratic coefficient gives $\frac{(1/m)(1/m-1)}{2}=\frac{1}{2}\cdot\frac{1}{m}\cdot\left(\frac{1}{m}-1\right)=\frac{1}{2}\cdot\frac{1}{m}\cdot\left(\frac{1-m}{m}\right)=\frac{1-m}{2m^2}$ so that $(x+1)^{1/m}=1+\frac{1}{m}x+\frac{1-m}{2m^2}x^2+\cdots$. If $x$ is close to zero (the center of the series), then the second-degree partial sum of this series will give good approximations to the actual value of $(1+x)^{1/m}$. (If $x$ is close to zero, $x^3$ and higher-order powers of $x$ are *really* small.) This means that the given polynomial is a good approximation for $(x+1)^{1/m}$.

19. We can still use our tableau, though we need to modify it a bit.

$$\begin{array}{lcl} f(0)=3 & \to & \frac{3}{0!}=3 \\ f'(0)=\frac{(-1)^1\cdot(1+1)}{1^2}=-2 & \to & \frac{-2}{1!}=-2 \\ f''(0)=\frac{(-1)^2\cdot(2+1)}{2^2}=\frac{3}{4} & \to & \frac{3/4}{2!}=\frac{3}{8} \\ f'''(0)=\frac{(-1)^3\cdot(3+1)}{3^2}=\frac{-4}{9} & \to & \frac{-4/9}{3!}=\frac{-2}{27} \end{array}$$

$f(x)=3-2x+\frac{3}{8}x^2-\frac{2}{27}x^3+\cdots+\frac{(-1)^n\cdot(n+1)}{n^2\cdot n!}x^n+\cdots$ (Don't forget to divide by $n$!.)

$\lim\limits_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim\limits_{n\to\infty}\left|\frac{(-1)^{n+1}\cdot(n+2)\cdot x^{n+1}}{(n+1)^2\cdot(n+1)!}\cdot\frac{n^2\cdot n!}{(-1)^n\cdot(n+1)\cdot x^n}\right|=\lim\limits_{n\to\infty}\left|\frac{n+2}{n+1}\cdot\frac{n^2}{(n+1)^2}\cdot\frac{n!}{(n+1)!}\cdot x\right|=\lim\limits_{n\to\infty}\left|\frac{x}{n+1}\right|=0<1$. This series converges for all $x$.

Interval of convergence: -∞ < $x$ ∞.

20. We again use the tableau to organize our computation of the coefficients.

$$g(0)=\frac{3^0+0^2}{2^0}=1 \quad\rightarrow\quad \frac{1}{0!}=1$$
$$g'(0)=\frac{3^1+1^2}{2^1}=2 \quad\rightarrow\quad \frac{2}{1!}=2$$
$$g''(0)=\frac{3^2+2^2}{2^2}=\frac{13}{4} \quad\rightarrow\quad \frac{13/4}{2!}=\frac{13}{8}$$
$$g'''(0)=\frac{3^3+3^2}{2^3}=\frac{9}{2} \quad\rightarrow\quad \frac{9/2}{3!}=\frac{3}{4}$$

$g(x)=1+2x+\frac{13}{8}x^2+\frac{3}{4}x^3+\cdots+\frac{3^n+n^2}{2^n\cdot n!}x^n+\cdots$

$\lim\limits_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim\limits_{n\to\infty}\left|\frac{\left(3^{n+1}+(n+1)^2\right)x^{n+1}}{2^{n+1}\cdot(n+1)!}\cdot\frac{2^n\cdot n!}{\left(3^n+n^2\right)x^n}\right|=\lim\limits_{n\to\infty}\left|\frac{3^{n+1}+(n+1)^2}{3^n+n^2}\cdot\frac{2^n}{2^{n+1}}\cdot\frac{1}{n+1}\cdot x\right|=0<1$; the power series converges for all real numbers.

Interval of convergence: $-\infty<x<\infty$

21. We begin with the tableau.

$$h(2)=\frac{0!}{0+1}=1 \quad\rightarrow\quad \frac{1}{0!}=1$$
$$h'(2)=\frac{1!}{1+1}=\frac{1}{2} \quad\rightarrow\quad \frac{1/2}{1!}=\frac{1}{2}$$
$$h''(2)=\frac{2!}{2+1}=\frac{2}{3} \quad\rightarrow\quad \frac{2/3}{2!}=\frac{1}{3}$$
$$h'''(2)=\frac{3!}{3+1}=\frac{6}{4} \quad\rightarrow\quad \frac{6/4}{3!}=\frac{1}{4}$$

$h(x)=1+\frac{1}{2}(x-2)+\frac{1}{3}(x-2)^2+\frac{1}{4}(x-2)^3+\cdots+\frac{1}{n+1}(x-2)^n+\cdots$

$\lim\limits_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim\limits_{n\to\infty}\left|\frac{(x-2)^{n+1}}{n+2}\cdot\frac{n+1}{(x-2)^n}\right|=\lim\limits_{n\to\infty}\left|\frac{n+1}{n+2}\cdot(x-2)\right|=|x-2|$

$|x-2|<1\Rightarrow -1<x-2<1\Rightarrow 1<x<3$

$x=3$: $\sum\limits_{n=0}^{\infty}\frac{1}{n+1}(3-2)^n=\sum\limits_{n=0}^{\infty}\frac{1}{n+1}=\sum\limits_{n=1}^{\infty}\frac{1}{n}$. This is the harmonic series. It diverges.

$x=1$: $\sum\limits_{n=0}^{\infty}\frac{1}{n+1}(1-2)^n=\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{n+1}=\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$. This is the alternating harmonic series. It converges.

Interval of convergence: $1\le x<3$

22. a. $f(x)=8+\sum\limits_{n=2}^{\infty}\frac{(-1)^n n!}{\left(2^n+n^2\right)n!}\cdot(x+1)^n=8+\sum\limits_{n=2}^{\infty}\frac{(-1)^n(x+1)^n}{2^n+n^2}=8+\frac{1}{8}(x+1)^2-\frac{1}{17}(x+1)^3+\frac{1}{32}(x+1)^4+\cdots$.

$\lim\limits_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim\limits_{n\to\infty}\left|\frac{(-1)^{n+1}\cdot(x+1)^{n+1}}{2^{n+1}+(n+1)^2}\cdot\frac{2^n+n^2}{(-1)^n\cdot(x+1)^n}\right|=\lim\limits_{n\to\infty}\left|\frac{2^n+n^2}{2^{n+1}+(n+1)^2}\cdot(x+1)\right|=\frac{1}{2}|x+1|$

$\frac{1}{2}|x+1|<1\Rightarrow -1<\frac{1}{2}(x+1)<1\Rightarrow -2<x+1<2\Rightarrow -3<x<1$.

$x$ = -3: $\sum\limits_{n=2}^{\infty}\frac{(-1)^n(-3+1)^n}{2^n+n^2}=\sum\limits_{n=2}^{\infty}\frac{(-1)^n\cdot(-2)^n}{2^n+n^2}=\sum\limits_{n=2}^{\infty}\frac{2^n}{2^n+n^2}$. $\lim\limits_{n\to\infty}\frac{2^n}{2^n+n^2}=1\ne 0$. This series fails the $n^{\text{th}}$ term test. It diverges.

$x$ = 1: $\sum\limits_{n=2}^{\infty}\frac{(-1)^n\cdot 2^n}{2^n+n^2}$. This series is the same as the previous one, but alternating. It still fails the $n^{\text{th}}$ term test.

Interval of convergence: -3 < $x$ < 1

b. $x$ = -1 is a critical point of the function since $f'(-1)=0$. From the second-degree term of the Taylor series, we see that $\frac{f''(-1)}{2!}=\frac{1}{8}$. Therefore $f''(-1)>0$. The function has a local minimum at $x=-1$ by the second derivative test.

c. $g(x)=f(x-1)=8+\frac{1}{8}\left((x-1)+1\right)^2-\frac{1}{17}\left((x-1)+1\right)^3+\cdots$

$g(x)=8+\frac{1}{8}x^2-\frac{1}{17}x^3+\cdots+\frac{(-1)^n}{2^n+n^2}x^n+\cdots$

$h(x)=g\left(x^2\right)=8+\tfrac{1}{8}\left(x^2\right)^2-\tfrac{1}{17}\left(x^2\right)^3+\cdots+\tfrac{(-1)^n}{2^n+n^2}\left(x^2\right)^n+\cdots$

$h(x)=8+\tfrac{1}{8}x^4-\tfrac{1}{17}x^6+\cdots+\tfrac{(-1)^n}{2^n+n^2}x^{2n}+\cdots$

23. Since the coefficient are given by $\frac{2^n\cdot(n+1)}{n!}$, we know that $\frac{f^{(n)}(0)}{n!}=\frac{2^n\cdot(n+1)}{n!}$. Simplifying gives $f^{(n)}(0)=2^n\cdot(n+1)$. $f^{(4)}(0)=2^4\cdot(4+1)=80$.
    Since the series we have is a Maclaurin series, we cannot determine values of the derivatives of $f$ at any $x$-value other than 0 exactly. We cannot determine $f^{(4)}(1)$ exactly.
24. The coefficient of $(x-3)^5$ in the Taylor series will be given by $\frac{g^{(5)}(3)}{5!}$. Therefore $\frac{g^{(5)}(3)}{5!}=\frac{(-1)^5\cdot 3^5}{5^2+1}=\frac{-243}{26}$.
    It follows that $g^{(5)}(3)=\frac{14580}{13}$.
25. $\cos(x)=1-\tfrac{1}{2!}x^2+\tfrac{1}{4!}x^4-\cdots$

    $\cos\left(x^3\right)=1-\tfrac{1}{2!}\left(x^3\right)^2+\tfrac{1}{4!}\left(x^3\right)^4-\cdots$

    $=1-\tfrac{1}{2!}x^6+\tfrac{1}{24}x^{12}-\cdots$

    The desired coefficient is 1/24.
26. $e^x=1+x+\tfrac{1}{2!}x^2+\tfrac{1}{3!}x^3+\tfrac{1}{4!}x^4+\cdots$

    $x^2e^x=x^2\left(1+x+\tfrac{1}{2!}x^2+\tfrac{1}{3!}x^3+\tfrac{1}{4!}x^4+\cdots\right)$

    $=x^2+x^3+\tfrac{1}{2!}x^4+\tfrac{1}{3!}x^5+\tfrac{1}{4!}x^6+\cdots$

    The desired coefficient is 1/4! = 1/24.
27. We will take the indirect approach suggested in the problem. We begin with

$$f(x)=\tfrac{x}{1-x-x^2}$$

$$\left(1-x-x^2\right)f(x)=x.$$

We now assume that there is a Maclaurin series for $f(x)$.

$$\left(1-x-x^2\right)\left(c_0+c_1x+c_2x^2+c_3x^3+\cdots\right)=x$$

$$c_0+c_1x+c_2x^2+c_3x^3+\cdots-c_0x-c_1x^2-c_2x^3-\cdots-c_0x^2-c_1x^3-\cdots=x$$

$$c_0+\left(c_1-c_0\right)x+\left(c_2-c_1-c_0\right)x^2+\left(c_3-c_2-c_1\right)x^3+\cdots=x$$

For $n\geq 2$, $c_n-c_{n-1}-c_{n-2}$ is the coefficient of $x^n$ on the left side.

Equating coefficients of like powers, we see that

$$c_0=0$$

$$c_1-c_0=1$$

$$c_2-c_1-c_0=0$$

$$c_3-c_2-c_1=0$$

and, in general, $c_n-c_{n-1}-c_{n-2}=0$ for $n\geq 2$. Rewriting this, we have $c_n=c_{n-1}+c_{n-2}$.

Since $c_0=0$, $c_1=1$. Now $c_2-(1-0)=0$, so $c_2=1$. From here and the fact that $c_n=c_{n-1}+c_{n-2}$, we can generate arbitrarily many coefficients. They are simply the Fibonacci numbers.

$f(x)=1x+1x^2+2x^3+3x^4+5x^5+\cdots+F_nx^n+\cdots$ where $F_n$ represents the $n^{\text{th}}$ Fibonacci number.

28. a. Rather than start from scratch, let's use some log properties.

$$f(x)=\ln\left(\tfrac{1+x}{1-x}\right)=\ln(1+x)-\ln(1-x)$$
$$=\left(x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\tfrac{1}{5}x^5-\cdots\right)-\left(-x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3-\tfrac{1}{4}x^4-\tfrac{1}{5}x^5-\cdots\right)$$
$$=2x+\tfrac{2}{3}x^3+\tfrac{2}{5}x^5+\cdots=2\cdot\sum_{n=0}^{\infty}\tfrac{x^{2n+1}}{2n+1}$$

b. $\lim\limits_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim\limits_{n\to\infty}\left|\frac{x^{2n+3}}{2n+3}\cdot\frac{2n+1}{x^{2n+1}}\right|=\lim\limits_{n\to\infty}\left|\frac{2n+1}{2n+3}\cdot x^2\right|=x^2$

$x^2<1\Rightarrow|x|<1\Rightarrow -1<x<1$

$x = 1$: $2\cdot\sum\limits_{n=0}^{\infty}\frac{1^{2n+1}}{2n+1}=2\cdot\sum\limits_{n=0}^{\infty}\frac{1}{2n+1}$. This series diverges. (See Section 7, Problem 19.)

$x = -1$: $2\cdot\sum\limits_{n=0}^{\infty}\frac{(-1)^{2n+1}}{2n+1}=2\cdot\sum\limits_{n=0}^{\infty}\frac{-1}{2n+1}=-2\cdot\sum\limits_{n=0}^{\infty}\frac{1}{2n+1}$. Still diverges.

Interval of convergence: $-1 < x < 1$

c. Suppose $\alpha$ is a given positive number. We set $\alpha=\frac{1+x}{1-x}$ and solve for *x*.

$$\alpha=\tfrac{1+x}{1-x}$$
$$\alpha(1-x)=1+x$$
$$\alpha-\alpha x=1+x$$
$$\alpha-1=\alpha x+x$$
$$\alpha-1=x(\alpha+1)$$
$$x=\tfrac{\alpha-1}{\alpha+1}$$

The algebra above shows that for any positive number $\alpha$ (actually any number $\alpha$, though we want to respect the domain of the natural logarithm function), we can find an *x*-value such that $\alpha=\frac{1+x}{1-x}$. Furthermore, if $\alpha>0$, then $\frac{\alpha-1}{\alpha+1}$ lies between -1 and 1. (A rigorous proof of this fact is not terribly necessary in this context. Look at a graph of $f(\alpha)=\frac{\alpha-1}{\alpha+1}$ if you are not convinced of this.) Thus the desired *x*-value is in the interval of convergence of the series for $f(x)$.

d. If $\alpha=15$, then $x=\frac{\alpha-1}{\alpha+1}=\frac{15-1}{15+1}=\frac{14}{16}=\frac{7}{8}$.

$\ln\left(\frac{1+\frac{7}{8}}{1-\frac{7}{8}}\right)\approx 2\left(\tfrac{7}{8}\right)+\tfrac{2}{3}\left(\tfrac{7}{8}\right)^3+\tfrac{2}{5}\left(\tfrac{7}{8}\right)^5+\tfrac{2}{7}\left(\tfrac{7}{8}\right)^7=2.51398$ (Calculator value for ln(15): 2.70805)

29. a. First note that $\sinh(0)=\frac{e^0-e^{-0}}{2}=0$ while $\cosh(0)=\frac{e^0+e^{-0}}{2}=1$. Furthermore, $\frac{d}{dx}(\sinh x)=\frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right)=\frac{e^x+e^{-x}}{2}=\cosh x$ while $\frac{d}{dx}(\cosh x)=\frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right)=\frac{e^x-e^{-x}}{2}=\sinh x$. Now we attack the hyperbolic sine function.

$$\begin{array}{lclcl}
f(x)=\sinh x & \to & f(0)=0 & \to & 0\\
f'(x)=\cosh x & \to & f'(0)=1 & \to & 1\\
f''(x)=\sinh x & \to & f''(0)=0 & \to & 0\\
f'''(x)=\cosh x & \to & f'''(0)=1 & \to & \frac{1}{3!}\\
f^{(4)}(x)=\sinh x & \to & f^{(4)}(0)=0 & \to & 0\\
f^{(5)}(x)=\cosh x & \to & f^{(5)}(0)=1 & \to & \frac{1}{5!}
\end{array}$$

From this we see that $\sinh(x)=x+\frac{1}{3!}x^3+\frac{1}{5!}x^5+\cdots+\frac{1}{(2n+1)!}x^{2n+1}+\cdots$. Now for the hyperbolic cosine function.

$$\begin{array}{lclcl} f(x)=\cosh x & \to & f(0)=1 & \to & 1 \\ f'(x)=\sinh x & \to & f'(0)=0 & \to & 0 \\ f''(x)=\cosh x & \to & f''(0)=1 & \to & \frac{1}{2!} \\ f'''(x)=\sinh x & \to & f'''(0)=0 & \to & 0 \\ f^{(4)}(x)=\cosh x & \to & f^{(4)}(0)=1 & \to & \frac{1}{4!} \\ f^{(5)}(x)=\sinh x & \to & f^{(5)}(0)=0 & \to & 0 \end{array}$$

$\cosh(x)=1+\frac{1}{2!}x^2+\frac{1}{4!}x^4+\cdots+\frac{1}{(2n)!}x^{2n}+\cdots.$

b. $\cosh(ix)=1+\frac{1}{2!}(ix)^2+\frac{1}{4!}(ix)^4+\frac{1}{6!}(ix)^6+\cdots$

$=1+\frac{1}{2!}i^2x^2+\frac{1}{4!}i^4x^4+\frac{1}{6!}i^6x^6+\cdots$

$=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots=\cos(x)$

$\sinh(ix)=ix+\frac{1}{3!}(ix)^3+\frac{1}{5!}(ix)^5+\frac{1}{7!}(ix)^7+\cdots$

$=ix-\frac{1}{3!}ix^3+\frac{1}{5!}ix^5-\frac{1}{7!}ix^7+\cdots$

$=i\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots\right)=i\sin x$

$-i\sinh(ix)=-i\cdot i\sin x$

$=-i^2\sin x=\sin x$

30. $\lim\limits_{x\to 0}\frac{\tan^{-1}x}{x}=\lim\limits_{x\to 0}\frac{x-\frac{1}{3}x^3+\frac{1}{5}x^5-\cdots}{x}=\lim\limits_{x\to 0}\left(1-\frac{1}{3}x^2+\frac{1}{5}x^4-\cdots\right)=1$

Also, by l'Hospital's rule, $\lim\limits_{x\to 0}\frac{\tan^{-1}x}{x}=\lim\limits_{x\to 0}\frac{\frac{1}{1+x^2}}{1}=1$.

31. $\lim\limits_{x\to 0}\frac{\cos(x^2)-1}{x^4}=\lim\limits_{x\to 0}\frac{\left(1-\frac{1}{2!}x^4+\frac{1}{4!}x^8-\cdots\right)-1}{x^4}=\lim\limits_{x\to 0}\frac{\frac{-1}{2!}x^4+\frac{1}{4!}x^8-\cdots}{x^4}=\lim\limits_{x\to 0}\left(\frac{-1}{2!}+\frac{1}{4!}x^4-\cdots\right)=\frac{-1}{2}$

Also, by l'Hospital's rule, $\lim\limits_{x\to 0}\frac{\cos(x^2)-1}{x^4}=\lim\limits_{x\to 0}\frac{-2x\cdot\sin(x^2)}{4x^3}=\lim\limits_{x\to 0}\frac{-\sin(x^2)}{2x^2}=\lim\limits_{x\to 0}\frac{-2x\cdot\cos(x^2)}{4x}=\lim\limits_{x\to 0}\frac{-1}{2}\cos(x^2)=\frac{-1}{2}$.

32. $\lim\limits_{x\to 0}\frac{1-x-e^x}{\sin x}=\lim\limits_{x\to 0}\frac{1-x-\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots\right)}{x-\frac{1}{3!}x^3+\cdots}=\lim\limits_{x\to 0}\frac{-2x-\frac{1}{2!}x^2-\frac{1}{3!}x^3-\cdots}{x-\frac{1}{3!}x^3+\cdots}=\lim\limits_{x\to 0}\frac{-2-\frac{1}{2!}x-\frac{1}{3!}x^2-\cdots}{1-\frac{1}{3!}x^2+\cdots}=-2$

Verifying by l'Hospital's rule… $\lim\limits_{x\to 0}\frac{1-x-e^x}{\sin x}=\lim\limits_{x\to 0}\frac{-1-e^x}{\cos x}=-2$.

33. $\int\cos(x^2)\,dx=\int\left(1-\frac{1}{2!}(x^2)^2+\frac{1}{4!}(x^2)^4-\frac{1}{6!}(x^2)^6+\cdots\right)dx=\int\left(1-\frac{1}{2!}x^4+\frac{1}{4!}x^8-\frac{1}{6!}x^{12}+\cdots\right)dx$

$=C+x-\frac{1}{5\cdot 2!}x^5+\frac{1}{9\cdot 4!}x^9+\frac{1}{13\cdot 6!}x^{13}+\cdots=C+\sum_{n=0}^{\infty}\frac{(-1)^n}{(4n+1)\cdot(2n)!}x^{4n+1}$

34. $\int\frac{e^x-1}{x}\,dx=\int\frac{\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\cdots\right)-1}{x}\,dx=\int\frac{x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\cdots}{x}\,dx=\int\left(1+\frac{1}{2!}x+\frac{1}{3!}x^2+\frac{1}{4!}x^3+\cdots\right)dx$

$=C+x+\frac{1}{2\cdot 2!}x^2+\frac{1}{3\cdot 3!}x^3+\frac{1}{4\cdot 4!}x^4+\cdots=C+\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}x^n$

35. $\int\frac{e^x}{x}\,dx=\int\frac{1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots}{x}\,dx=\int\left(\frac{1}{x}+1+\frac{1}{2!}x+\frac{1}{3!}x^2+\cdots\right)dx$

$=C+\ln|x|+x+\frac{1}{2\cdot 2!}x^2+\frac{1}{3\cdot 3!}x^3\cdots=C+\ln|x|+\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}x^n$

36. $\sqrt{1+x^3}$ is binomial with $k$ = 1/2.

$\sqrt{1+x}=(1+x)^{1/2}=1+\frac{1}{2}x+\frac{(1/2)(-1/2)}{2!}x^2+\frac{(1/2)(-1/2)(-3/2)}{3!}x^3+\cdots$

$\sqrt{1+x^3}=\left(1+x^3\right)^{1/2}=1+\frac{1}{2}x^3+\frac{(1/2)(-1/2)}{2!}x^6+\frac{(1/2)(-1/2)(-3/2)}{3!}x^9+\cdots$

$$\int\left(1+x^3\right)^{1/2}dx=\int\left(1+\tfrac{1}{2}x^3-\tfrac{1}{8}x^6+\tfrac{1}{16}x^9+\cdots\right)dx=C+x+\tfrac{1}{8}x^4-\tfrac{1}{56}x^7+\tfrac{1}{160}x^{10}+\cdots$$

37. $$\int_0^1\cos\left(x^2\right)dx=\int_0^1\left(1-\tfrac{1}{2!}x^4+\tfrac{1}{4!}x^8-\tfrac{1}{6!}x^{12}+\cdots\right)dx=\left(x-\tfrac{1}{5\cdot 2!}x^5+\tfrac{1}{9\cdot 4!}x^9-\tfrac{1}{13\cdot 6!}x^{13}+\cdots\right)\Big|_0^1$$

$$=1-\tfrac{1}{5\cdot 2!}+\tfrac{1}{9\cdot 4!}-\tfrac{1}{13\cdot 6!}+\cdots=\sum_{n=0}^{\infty}\tfrac{(-1)^n}{(4n+1)\cdot(2n)!}$$

$$\int_0^1\cos\left(x^2\right)dx\approx 1-\tfrac{1}{5\cdot 2!}+\tfrac{1}{9\cdot 4!}=0.9046296$$

The series that gives this integral is alternating, so the maximum error is the first omitted term, in this case $\tfrac{1}{13\cdot 6!}=0.0001068$.

38. $$\int_0^2\frac{dx}{e^x}=\int_0^2 e^{-x}dx=\int_0^2\left(1-x+\tfrac{1}{2!}x^2-\tfrac{1}{3!}x^3+\cdots\right)dx=\left(x-\tfrac{1}{2}x^2+\tfrac{1}{3!}x^3-\tfrac{1}{4!}x^4+\cdots\right)\Big|_0^2$$

$$=2-2+\tfrac{8}{3!}-\tfrac{16}{4!}+\cdots=\sum_{n=1}^{\infty}\tfrac{(-1)^{n+1}\cdot 2^n}{n!}$$

$$\int_0^2\frac{dx}{e^x}\approx 2-2+\tfrac{8}{3!}=\tfrac{4}{3}$$

The series that gives this integral is alternating, so the maximum error is the first omitted term, in this case $\tfrac{16}{4!}=\tfrac{16}{24}=\tfrac{2}{3}$. That's a lot of error; we should use more terms in the approximation.

Of note, this integral can be evaluated directly, using elementary methods.

$\int_0^2 e^{-x}dx=-e^{-x}\Big|_0^2=-e^{-2}+e^0=1-\tfrac{1}{e^2}$. From this problem, then, we can conclude that $\sum_{n=1}^{\infty}\tfrac{(-1)^{n+1}\cdot 2^n}{n!}=1-\tfrac{1}{e^2}$.

39. $$\int_0^1\tfrac{\cos(3x)-1}{x}dx=\int_0^1\tfrac{\left(1-\frac{1}{2!}(3x)^2+\frac{1}{4!}(3x)^4-\frac{1}{6!}(3x)^6+\cdots\right)-1}{x}dx=\int_0^1\tfrac{-\frac{1}{2!}3^2\cdot x^2+\frac{1}{4!}3^4\cdot x^4-\frac{1}{6!}3^6\cdot x^6+\cdots}{x}dx$$

$$=\int_0^1\left(\tfrac{-1}{2!}\cdot 3^2\cdot x+\tfrac{1}{4!}\cdot 3^4\cdot x^3-\tfrac{1}{6!}3^6\cdot x^5+\cdots\right)dx=\left(\tfrac{-3^2}{2\cdot 2!}x^2+\tfrac{3^4}{4\cdot 4!}x^4-\tfrac{3^6}{6\cdot 6!}x^6+\cdots\right)\Big|_0^1$$

$$=\tfrac{-3^2}{2\cdot 2!}+\tfrac{3^4}{4\cdot 4!}-\tfrac{3^6}{6\cdot 6!}+\cdots=\sum_{n=1}^{\infty}\tfrac{(-1)^n\cdot 3^{2n}}{(2n)\cdot(2n)!}$$

$$\int_0^1\tfrac{\cos(3x)-1}{x}dx\approx\tfrac{-3^2}{2\cdot 2!}+\tfrac{3^4}{4\cdot 4!}-\tfrac{3^6}{6\cdot 6!}=\tfrac{-63}{40}=-1.575$$

The series that gives this integral is alternating, so the maximum error is the first omitted term, in this case $\tfrac{3^8}{8\cdot 8!}=0.02034$.

40. a. We will obtain a series for $\tfrac{\sin t}{t}$, and integrate term by term.

$$\sin t=t-\tfrac{1}{3!}t^3+\tfrac{1}{5!}t^5-\tfrac{1}{7!}t^7+\cdots$$

$$\tfrac{\sin t}{t}=\tfrac{t-\frac{1}{3!}t^3+\frac{1}{5!}t^5-\frac{1}{7!}t^7+\cdots}{t}=1-\tfrac{1}{3!}t^2+\tfrac{1}{5!}t^4-\tfrac{1}{7!}t^6+\cdots$$

$$\int_0^x\tfrac{\sin t}{t}dt=\int_0^x\left(1-\tfrac{1}{3!}t^2+\tfrac{1}{5!}t^4-\tfrac{1}{7!}t^6+\cdots\right)dt=\left(t-\tfrac{1}{3\cdot 3!}t^3+\tfrac{1}{5\cdot 5!}t^5-\tfrac{1}{7\cdot 7!}t^7+\cdots\right)\Big|_0^x$$

$$=x-\tfrac{1}{3\cdot 3!}x^3+\tfrac{1}{5\cdot 5!}x^5-\tfrac{1}{7\cdot 7!}x^7+\cdots=\sum_{n=0}^{\infty}\tfrac{(-1)^n}{(2n+1)\cdot(2n+1)!}x^{2n+1}$$

b. We simply plug in 1 for $x$ in the series from part (a).

$$\mathrm{Si}(x)=\sum_{n=0}^{\infty}\tfrac{(-1)^n}{(2n+1)\cdot(2n+1)!}x^{2n+1}$$

$$\mathrm{Si}(1)=\sum_{n=0}^{\infty}\tfrac{(-1)^n}{(2n+1)\cdot(2n+1)!}(1)^{2n+1}=\sum_{n=0}^{\infty}\tfrac{(-1)^n}{(2n+1)\cdot(2n+1)!}$$

c. Since the series for Si(1) is alternating, we need to find when $\left|a_{n+1}\right|<10^{-6}$. Solving $\tfrac{1}{(2n+1)\cdot(2n+1)!}<10^{-6}$ with a table shows that the desired accuracy is reached as long as $n\geq 4$. Since the summation starts with $n=0$, including terms up to $n=4$ means a total of five terms.

41. a. We will obtain a series for $e^{-t^2}$ and integrate term by term.

$$e^t=1+t+\tfrac{1}{2!}t^2+\tfrac{1}{3!}t^3+\cdots$$

$$e^{-t^2}=1+\left(-t^2\right)+\tfrac{1}{2!}\left(-t^2\right)^2+\tfrac{1}{3!}\left(-t^2\right)^3+\cdots=1-t^2+\tfrac{1}{2!}t^4-\tfrac{1}{3!}t^6+\cdots$$

$$\int_0^x e^{-t^2}\,dt=\int_0^x\left(1-t^2+\tfrac{1}{2!}t^4-\tfrac{1}{3!}t^6+\cdots\right)dt=\left(t-\tfrac{1}{3}t^3+\tfrac{1}{5\cdot 2!}t^5-\tfrac{1}{7\cdot 3!}t^7+\cdots\right)\Big|_0^x$$

$$=x-\tfrac{1}{3}x^3+\tfrac{1}{5\cdot 2!}x^5-\tfrac{1}{7\cdot 3!}x^7+\cdots=\sum_{n=0}^{\infty}\tfrac{(-1)^n}{(2n+1)\cdot n!}x^{2n+1}$$

$$\tfrac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt=\tfrac{2}{\sqrt{\pi}}\cdot\sum_{n=0}^{\infty}\tfrac{(-1)^n}{(2n+1)\cdot n!}x^{2n+1}$$

b. We cam simply substitute $\tfrac{x}{\sqrt{2}}$ for $x$ in the series for erf($x$).

$$\mathrm{erf}(x)=\tfrac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt=\tfrac{2}{\sqrt{\pi}}\cdot\sum_{n=0}^{\infty}\tfrac{(-1)^n}{(2n+1)\cdot n!}x^{2n+1}$$

$$\mathrm{erf}\left(\tfrac{x}{\sqrt{2}}\right)=\tfrac{2}{\sqrt{\pi}}\int_0^{x/\sqrt{2}} e^{-t^2}\,dt=\tfrac{2}{\sqrt{\pi}}\cdot\sum_{n=0}^{\infty}\tfrac{(-1)^n}{(2n+1)\cdot n!}\left(\tfrac{x}{\sqrt{2}}\right)^{2n+1}=\tfrac{2}{\sqrt{\pi}}\cdot\sum_{n=0}^{\infty}\tfrac{(-1)^n}{(2n+1)\cdot n!}\cdot\tfrac{1}{\sqrt{2}^{2n+1}}\cdot x^{2n+1}$$

c. $$\mathrm{erf}\left(\tfrac{1}{\sqrt{2}}\right)=\tfrac{2}{\sqrt{\pi}}\cdot\sum_{n=0}^{\infty}\tfrac{(-1)^n}{(2n+1)\cdot n!}\cdot\tfrac{1}{\sqrt{2}^{2n+1}}\approx\tfrac{2}{\sqrt{\pi}}\left(\tfrac{1}{1\cdot 0!\cdot\sqrt{2}^1}-\tfrac{1}{3\cdot 1!\cdot\sqrt{2}^3}+\tfrac{1}{5\cdot 2!\cdot\sqrt{2}^5}-\tfrac{1}{7\cdot 3!\cdot\sqrt{2}^7}\right)=0.6825$$

Since this series is alternating, the error is no more than the next omitted term. In this case, that is $\tfrac{2}{\sqrt{\pi}}\tfrac{1}{9\cdot 4!\cdot\sqrt{2}^9}=0.0002309$.

42. a. This series is geometric with initial term 1 and common ratio 3$x$. $f(x)=\tfrac{1}{1-3x}$.

b. $f(x)=\arctan x$

c. The odd factorials suggest the sine series. However, the powers don't match the factorials; they are each 1 degree too small, suggesting a division by $x$. Also, we are missing what used to be the linear term of the sine series. Putting it all together, we have $f(x)=\tfrac{\sin x-x}{x}$.

d. The even factorials suggest the cosine series. The powers of 5 in the numerators suggest that $x$ has been replaced by 5$x$. $f(x)=\cos(5x)$.

43. a. $f(x)=\ln(1+x)$

b. This looks like the sine series, except for the coefficients, which are odd powers of 2. $f(x)=\sin(2x)$.

c. This series is geometric with initial term $8x^2$ and common ratio $\tfrac{-1}{2}x^2$. $f(x)=\tfrac{8x^2}{1-\left(\tfrac{-1}{2}x^2\right)}=\tfrac{8x^2}{1+\tfrac{1}{2}x^2}=\tfrac{16x^2}{2+x^2}$.

d. This is the series for $e^x$ except that it is missing the constant term and the terms alternate. The alternation suggests $e^{-x}$, but $e^{-x}=1-x+\tfrac{1}{2!}x^2-\tfrac{1}{3!}x^3+\cdots$. We can fix both the constant term and the sign issue in one fell swoop: $f(x)=1-e^{-x}$.

44. a. This would be the cosine series (as given away by the even factorial denominators), except that the degree of every term is three more than it should be. $f(x) = x^3 \cdot \cos(x)$.

b. This would be the sine series (as given away by the odd factorial denominators), except that the degree of every term is twice what it should be. $f(x) = \sin\left(x^2\right)$.

c. At first glance, this looks like some variation on the exponential series, but the signs are all out of whack. If you look at every *other* term, you will see either the sine series or the cosine series, depending on whether you start with the 1 or the $x$. $f(x) = \cos(x) + \sin(x)$.

d. Let's break up those grouped terms.

$$1+\left(1-\tfrac{1}{2!}\right)x^2+\left(\tfrac{1}{2!}+\tfrac{1}{4!}\right)x^4+\left(\tfrac{1}{3!}-\tfrac{1}{6!}\right)x^6+\cdots=\left(1+x^2+\tfrac{1}{2!}x^4+\tfrac{1}{3!}x^6+\cdots\right)+\left(-\tfrac{1}{2!}x^2+\tfrac{1}{4}x^4-\tfrac{1}{6!}x^6+\cdots\right)$$

The first grouping is the series for $e^{x^2}$. The second grouping is the series for cos($x$), missing the constant term. $f(x) = e^{x^2} + \cos(x) - 1$.

45. The radius of convergence will be as big as it can be before running into a singularity of the function, in this case at $\pm\frac{\pi}{2}$. We cannot include $\pm\frac{\pi}{2}$ in the interval of convergence, since the function being modeled is undefined there.

Radius of convergence: $\frac{\pi}{2}$ Interval of convergence: $\frac{-\pi}{2} < x < \frac{\pi}{2}$

46. $f(x) = \frac{1}{1-x}$ is undefined at $x = 1$. Of the two possible centers, $x = -2$ is farther from $x = 0$. Therefore the series centered at $x = -2$ will have the larger radius of convergence (namely $R = 3$ compared to $R = 1$ for the series centered at $x = 0$).

47. a. To maximize the radius of convergence, we want the center to be as far as possible from the vertical asymptotes at $x = 3$ and $x = 4$. Therefore we should center the series at $x = 3.5$.

b. The largest subinterval of [-1, 5] that does not have a vertical asymptote in it is [1, 3], which is two units wide. If we center a series in the middle of this interval, at $x = 2$, then the radius of convergence will be 1. This is the largest radius of convergence we can manage for this function on [-1, 5].

48. $\tan x = \frac{\sin x}{\cos x} = \frac{x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots}{1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots}$

$$\begin{array}{r l}
 & x+\tfrac{1}{3}x^3+\tfrac{2}{15}x^5+\tfrac{17}{315}x^7+\cdots \\
1-\tfrac{1}{2}x^2+\tfrac{1}{24}x^4-\tfrac{1}{720}x^6+\cdots \big) & x-\tfrac{1}{6}x^3+\tfrac{1}{120}x^5-\tfrac{1}{5040}x^7+\cdots \\
 & \underline{x-\tfrac{1}{2}x^3+\tfrac{1}{24}x^5-\tfrac{1}{720}x^7+\cdots} \\
 & \tfrac{1}{3}x^3-\tfrac{1}{30}x^5+\tfrac{1}{840}x^7-\cdots \\
 & \underline{\tfrac{1}{3}x^3-\tfrac{1}{6}x^5+\tfrac{1}{72}x^7-\cdots} \\
 & \tfrac{2}{15}x^5-\tfrac{4}{315}x^7 \\
 & \underline{\tfrac{2}{15}x^5-\tfrac{1}{15}x^7} \\
 & \tfrac{17}{315}x^7
\end{array}$$

$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$

Because the tangent function has vertical asymptotes at $x = \pm\frac{\pi}{2}$, the interval of convergence of its Maclaurin series will be $\frac{-\pi}{2} < x < \frac{\pi}{2}$. Therefore the radius of convergence will be $\pi/2$.

49. First note that if $y = \frac{-1}{9} + \frac{x}{3} + \sum_{n=0}^{\infty}\frac{(-1)^n \cdot 2 \cdot (3x)^n}{n!}$, then $y' = 0 + \frac{1}{3} + \sum_{n=1}^{\infty}\frac{(-1)^n \cdot 2 \cdot n \cdot (3x)^{n-1} \cdot 3}{n!}$. We have started $n$ at 1 in $y'$ only because the summand vanishes when $n = 0$; there is no reason to have $n = 0$. Also be careful of

the chain rule. That is why there is an extra 3 in the numerator of $y'$. Partially cancelling the factorial with the $n$ in the numerator gives $y' = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 6 \cdot (3x)^{n-1}}{(n-1)!}$.

Since $y = \frac{-1}{9} + \frac{x}{3} + \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2 \cdot (3x)^n}{n!}$, $3y = \frac{-1}{3} + x + 3 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2 \cdot (3x)^n}{n!} = \frac{-1}{3} + x + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 6 \cdot (3x)^{n-1}}{(n-1)!}$. We have re-indexed this series to be consistent with the series for $y'$.

Now we add $y'$ to $3y$ to see if we really get $x$. If we do, the given series does satisfy the differential equation.

$$
\begin{aligned}
y' + 3y &= \left( \frac{1}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 6 \cdot (3x)^{n-1}}{(n-1)!} \right) + \left( \frac{-1}{3} + x + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 6 \cdot (3x)^{n-1}}{(n-1)!} \right) \\
&= \frac{1}{3} + \frac{-1}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 6 \cdot (3x)^{n-1}}{(n-1)!} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 6 \cdot (3x)^{n-1}}{(n-1)!} + x \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 6 \cdot (3x)^{n-1}}{(n-1)!} + \sum_{n=1}^{\infty} \frac{-(-1)^n \cdot 6 \cdot (3x)^{n-1}}{(n-1)!} + x \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 6 \cdot (3x)^{n-1}}{(n-1)!} - \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 6 \cdot (3x)^{n-1}}{(n-1)!} + x \\
&= x
\end{aligned}
$$

(Note the manipulation of $(-1)^{n-1}$ to give $-(-1)^n$ in the third line.) The given function does indeed solve the differential equation.

50. Since $y = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n)!}$, $y' = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot (2n-1) \cdot x^{2n-2}}{(2n)!}$. It follows that $xy' = x \cdot \sum_{n=1}^{\infty} \frac{(-1)^n \cdot (2n-1) \cdot x^{2n-2}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot (2n-1) \cdot x^{2n-1}}{(2n)!}$.

$$
\begin{aligned}
xy' + y &= \sum_{n=1}^{\infty} \frac{(-1)^n \cdot (2n-1) \cdot x^{2n-1}}{(2n)!} + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} \left( \frac{(-1)^n \cdot (2n-1) \cdot x^{2n-1}}{(2n)!} + \frac{(-1)^n x^{2n-1}}{(2n)!} \right) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \left( (2n-1)x^{2n-1} + x^{2n-1} \right) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} (2n-1+1)x^{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} (2n)x^{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} x^{2n-1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{-(-1)^n x^{2n+1}}{(2n+1)!} = -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\
&= -\sin x
\end{aligned}
$$

As in Problem 49, we've done some manipulation of $(-1)^{n+1}$ and some re-indexing to obtain our final result. As we can see, though, the function checks out, so it is indeed a solution to the differential equation.

51. I will give two solutions for this one: one less formal and the other more formal. We know that $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n \cdot n!} = 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \frac{1}{384}x^8 + \cdots$. Differentiating term by term gives $y' = x + \frac{1}{2}x^3 + \frac{1}{8}x^5 + \frac{1}{48}x^7 + \cdots$ and $y'' = 1 + \frac{3}{2}x^2 + \frac{5}{8}x^4 + \frac{7}{48}x^6 + \cdots$. Multiplying the series for $y'$ by $x$ gives $xy' = x^2 + \frac{1}{2}x^4 + \frac{1}{8}x^6 + \frac{1}{48}x^8 + \cdots$. Let's put it all together.

$$\begin{aligned} y'' &= \quad 1+\tfrac{3}{2}x^2+\tfrac{5}{8}x^4+\tfrac{7}{48}x^6+\cdots \\ -xy' &= \qquad\ -x^2-\tfrac{1}{2}x^4-\tfrac{1}{8}x^6-\cdots \\ \underline{-y} &= \underline{-1-\tfrac{1}{2}x^2-\tfrac{1}{8}x^4-\tfrac{1}{48}x^6-\cdots} \\ y''-xy'-y &= \quad 0+0x^2+0x^4+0x^6+\cdots=0 \end{aligned}$$

So we see that this all checks out. The given function solves the differential equation.

More formally, since $y=\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n\cdot n!}=1+\sum_{n=1}^{\infty}\frac{x^{2n}}{2^n\cdot n!}$, $y'=\sum_{n=1}^{\infty}\frac{(2n)x^{2n-1}}{2^n\cdot n!}$ and $y''=\sum_{n=1}^{\infty}\frac{(2n)(2n-1)x^{2n-2}}{2^n\cdot n!}$. From this, we have $xy'=x\cdot\sum_{n=1}^{\infty}\frac{(2n)x^{2n-1}}{2^n\cdot n!}=\sum_{n=1}^{\infty}\frac{(2n)x^{2n}}{2^n\cdot n!}$. Both $y$ and $xy'$ have general terms with $x^{2n}$. We can make sure that the same is true of $y''$ if we do some re-indexing.

$$y''=\sum_{n=1}^{\infty}\frac{(2n)(2n-1)x^{2n-2}}{2^n\cdot n!}=\sum_{n=0}^{\infty}\frac{(2n+2)(2n+1)x^{2n}}{2^{n+1}\cdot(n+1)!}=\sum_{n=0}^{\infty}\frac{2(n+1)(2n+1)x^{2n}}{2\cdot 2^n\cdot(n+1)\cdot n!}=\sum_{n=0}^{\infty}\frac{(2n+1)x^{2n}}{2^n\cdot n!}$$

We must be doing something right; we just produced a common denominator for all three series. However, the series for $y''$ starts at $n = 0$ while the others start at $n = 1$. That's easy to fix.

$$y''=\sum_{n=0}^{\infty}\frac{(2n+1)x^{2n}}{2^n\cdot n!}=1+\sum_{n=1}^{\infty}\frac{(2n+1)x^{2n}}{2^n\cdot n!}$$

Now we put it all together.

$$\begin{aligned} y''-xy'-y &= \left(1+\sum_{n=1}^{\infty}\frac{(2n+1)x^{2n}}{2^n\cdot n!}\right)-\sum_{n=1}^{\infty}\frac{(2n)x^{2n}}{2^n\cdot n!}-\left(1+\sum_{n=1}^{\infty}\frac{x^{2n}}{2^n\cdot n!}\right) \\ &= \sum_{n=1}^{\infty}\frac{(2n+1)x^{2n}}{2^n\cdot n!}-\sum_{n=1}^{\infty}\frac{(2n)x^{2n}}{2^n\cdot n!}-\sum_{n=1}^{\infty}\frac{x^{2n}}{2^n\cdot n!} \\ &= \sum_{n=1}^{\infty}\left(\frac{(2n+1)x^{2n}}{2^n\cdot n!}-\frac{(2n)x^{2n}}{2^n\cdot n!}-\frac{x^{2n}}{2^n\cdot n!}\right)=\sum_{n=1}^{\infty}\frac{(2n+1)x^{2n}-(2n)x^{2n}-x^{2n}}{2^n\cdot n!} \\ &= \sum_{n=1}^{\infty}\frac{(2n+1-2n-1)x^{2n}}{2^n\cdot n!}=0 \end{aligned}$$

Once again we see that the function $y$ does satisfy the given differential equation.

52. We begin with the informal approach to verifying the solution.

Let $y=J_0(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}\cdot(n!)^2}=1-\frac{1}{4}x^2+\frac{1}{64}x^4-\frac{1}{2304}x^6+\frac{1}{147456}x^8-\cdots$. Differentiating term by term gives $y'=\frac{-1}{2}x+\frac{1}{16}x^3-\frac{1}{384}x^5+\frac{1}{18432}x^7-\cdots$ and $y''=\frac{-1}{2}+\frac{3}{16}x^2-\frac{5}{384}x^4+\frac{7}{18432}x^6-\cdots$. Now we will form the products $x^2y''$, $xy'$, and $x^2y$ and add them up.

$$\begin{aligned} x^2y'' &= \tfrac{-1}{2}x^2+\tfrac{3}{16}x^4-\tfrac{5}{384}x^6+\tfrac{7}{18432}x^8-\cdots \\ xy' &= \tfrac{-1}{2}x^2+\tfrac{1}{16}x^4-\tfrac{1}{384}x^6+\tfrac{1}{18432}x^8-\cdots \\ \underline{x^2y} &= \quad \underline{x^2-\tfrac{1}{4}x^4+\ \tfrac{1}{64}x^6-\tfrac{1}{2304}x^8-\cdots} \\ x^2y''+xy'+x^2 &= \ 0x^2+0x^4+\ \ 0x^6+\ \ 0x^8+\cdots=0 \end{aligned}$$

So we see that the function checks out; it solves the differential equation.

Here is a more formal solution:

$y=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}\cdot(n!)^2}=1+\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}\cdot(n!)^2}$, which implies that $y'=\sum_{n=1}^{\infty}\frac{(-1)^n(2n)x^{2n-1}}{2^{2n}\cdot(n!)^2}$ and $y''=\sum_{n=1}^{\infty}\frac{(-1)^n(2n)(2n-1)x^{2n-2}}{2^{2n}\cdot(n!)^2}$. The relevant products are $x^2y=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{2^{2n}\cdot(n!)^2}$, $xy'=\sum_{n=1}^{\infty}\frac{(-1)^n(2n)x^{2n}}{2^{2n}\cdot(n!)^2}$, and $x^2y''=\sum_{n=1}^{\infty}\frac{(-1)^n(2n)(2n-1)x^{2n}}{2^{2n}\cdot(n!)^2}$. We'd like to add these three expressions, but the general term for $x^2y$ has a different degree and starting index value from the other two. We can fix both problems by re-indexing: $x^2y=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{2^{2n}\cdot(n!)^2}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{2^{2n-2}\cdot[(n-1)!]^2}$.

Now the index variable and power are right, but we have the wrong denominator. Let's bring in a common denominator in advance of the addition:

$$x^2y=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{2^{2n-2}\cdot[(n-1)!]^2}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{2^{2n-2}\cdot[(n-1)!]^2}\cdot\frac{2\cdot 2\cdot n\cdot n}{2\cdot 2\cdot n\cdot n}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n)(2n)x^{2n}}{2^{2n}\cdot(n!)^2}$$

Now let's add.

$$\begin{aligned}
x^2y+xy'+xy''&=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n)(2n)x^{2n}}{2^{2n}\cdot(n!)^2}+\sum_{n=1}^{\infty}\frac{(-1)^n(2n)x^{2n}}{2^{2n}\cdot(n!)^2}+\sum_{n=1}^{\infty}\frac{(-1)^n(2n)(2n-1)x^{2n}}{2^{2n}\cdot(n!)^2}\\
&=\sum_{n=1}^{\infty}\left[\frac{(-1)^{n-1}(2n)(2n)x^{2n}}{2^{2n}\cdot(n!)^2}+\frac{(-1)^n(2n)x^{2n}}{2^{2n}\cdot(n!)^2}+\frac{(-1)^n(2n)(2n-1)x^{2n}}{2^{2n}\cdot(n!)^2}\right]\\
&=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n)(2n)x^{2n}+(-1)^n(2n)x^{2n}+(-1)^n(2n)(2n-1)x^{2n}}{2^{2n}\cdot(n!)^2}\\
&=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n)(2n)x^{2n}+(-1)^n(2n)x^{2n}\cdot[1+(2n-1)]}{2^{2n}\cdot(n!)^2}\\
&=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n)(2n)x^{2n}+(-1)^n(2n)(2n)x^{2n}}{2^{2n}\cdot(n!)^2}=\sum_{n=1}^{\infty}\frac{(2n)(2n)x^{2n}\cdot\left[(-1)^{n-1}+(-1)^n\right]}{2^{2n}\cdot(n!)^2}\\
&=\sum_{n=1}^{\infty}0=0
\end{aligned}$$

And so the solution checks; the given function satisfies the differential equation.

53. We proceed as in Examples 5 and 6, assuming that there is a Maclaurin series solution and then solving for the coefficients. To this end, let $y=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+\cdots$.

$$\begin{aligned}
y&=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+\cdots+c_nx^n+\cdots\\
y'&=c_1+2c_2x+3c_3x^2+4c_4x^3+\cdots+nc_nx^{n-1}+\cdots\\
y''&=2c_2+6c_3x+12c_4x^2+\cdots+n(n-1)c_nx^{n-2}+\cdots
\end{aligned}$$

Now we have

$$xy'=c_1x+2c_2x^2+3c_3x^3+4c_4x^4+\cdots+nc_nx^n+\cdots,$$

so that

$$\begin{aligned}
xy'+y&=\left(c_1x+2c_2x^2+3c_3x^3+4c_4x^4+\cdots\right)+\left(c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+\cdots\right)\\
&=c_0+2c_1x+3c_2x^2+4c_3x^3+5c_4x^4+\cdots+(n+1)c_nx^n+\cdots.
\end{aligned}$$

Equating coefficients of $y''$ and $xy'+y$ indicates that

$$2c_2 = c_0$$
$$6c_3 = 2c_1$$
$$12c_4 = 3c_2$$
$$\vdots$$
$$n(n-1)c_n = (n-1)c_{n-2}.$$

Of course, the last equality can be simplified to $nc_n = c_{n-2}$ or $c_n = \frac{1}{n}c_{n-2}$. Now it's time to use our initial conditions. Since $y(0)=0$, $c_0$ must be 0. This implies that $c_2 = 0$, $c_4 = 0$, and in general $c_{2k} = 0$ where *k* is a non-negative integer. Since $y'(0)=1$, $c_1$ must be 1. This implies that $c_3 = \frac{1}{3}$, which in turn implies that $c_5 = \frac{1}{5}\cdot c_3 = \frac{1}{5}\cdot\frac{1}{3} = \frac{1}{15}$. Then $c_7 = \frac{1}{7}\cdot\frac{1}{15} = \frac{1}{105}$, and so forth.

The solution to the initial value problem is $y = x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \frac{1}{105}x^7 + \cdots$.

54. We begin by assuming there is a Maclaurin series solution so that

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots + c_nx^n + \cdots$$
$$y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots + nc_nx^{n-1} + \cdots$$
$$y'' = 2c_2 + 6c_3x + 12c_4x^2 + \cdots + n(n-1)c_nx^{n-2} + \cdots.$$

Given this, it follows that

$$x^3y = c_0x^3 + c_1x^4 + c_2x^5 + c_3x^6 + c_4x^7 + \cdots + c_nx^{n+3} + \cdots.$$

Equating like coefficients between $y''$ and $x^3y$ gives

$$2c_2 = 0$$
$$6c_3 = 0$$
$$12c_4 = 0$$
$$20c_5 = c_0$$
$$30c_6 = c_1$$
$$42c_7 = c_2$$
$$\vdots$$

In general, $n(n-1)c_n = c_{n-5}$ for $n \ge 5$. Let's use the initial conditions. $y(0)=1$ implies that $c_0 = 1$. From this it follows that $20c_5 = 1$ or $c_5 = \frac{1}{20}$. In turn, $10\cdot 9\cdot c_{10} = c_5 \Rightarrow c_{10} = \frac{1}{90}\cdot c_5 = \frac{1}{1800}$. $y'(0)=1$ implies that $c_1 = 1$. Therefore $30c_6 = 1$ or $c_6 = \frac{1}{30}$. In turn, $10\cdot 11\cdot c_{11} = c_6 \Rightarrow c_{11} = \frac{1}{110}\cdot\frac{1}{30} = \frac{1}{3300}$.

The solution to the initial value problem is $y = 1 + x + \frac{1}{20}x^5 + \frac{1}{30}x^6 + \frac{1}{1800}x^{10} + \frac{1}{3300}x^{11} + \cdots$

55. a. We begin by assuming a Maclaurin series solution such that

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots + c_nx^n + \cdots$$
$$y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots + nc_nx^{n-1} + \cdots$$
$$y'' = 2c_2 + 6c_3x + 12c_4x^2 + \cdots + n(n-1)c_nx^{n-2} + \cdots.$$

Rather than tackle the differential equation as presented, let's rewrite it to $y'' + \lambda y = 2xy'$. This keeps everything positive and allows us to use the trick of equating coefficients. We will need the products $\lambda y$ and $2xy'$.

$$\lambda y = \lambda c_0 + \lambda c_1x + \lambda c_2x^2 + \lambda c_3x^3 + \lambda c_4x^4 + \cdots + \lambda c_nx^n + \cdots$$
$$2xy' = 2c_1x + 4c_2x^2 + 6c_3x^3 + 8c_4x^4 + \cdots + 2nc_nx^n + \cdots$$

Combining $y''$ and $\lambda y$ gives

$$2c_2 + \lambda c_0 + (6c_3 + \lambda c_1)x + (12c_4 + \lambda c_2)x^2 + (20c_5 + \lambda c_3)x^3 + \cdots + (n(n-1)c_n + \lambda c_{n-2})x^{n-2} + \cdots$$

where the general term is valid for $n \geq 2$.
Equating like terms between $y'' + \lambda y$ and $2xy'$ gives

$$\begin{aligned}2c_2 + \lambda c_0 &= 0 \Rightarrow 2c_2 = -\lambda c_0 \\ 6c_3 + \lambda c_1 &= 2c_1 \Rightarrow 6c_3 = (2-\lambda)c_1 \\ 12c_4 + \lambda c_2 &= 4c_2 \Rightarrow 12c_4 = (4-\lambda)c_2 \\ 20c_5 + \lambda c_3 &= 6c_3 \Rightarrow 20c_5 = (6-\lambda)c_3\end{aligned}$$

and so forth.
Now we are ready for the initial conditions. Plugging into the series expressions for $y$ and $y'$, we see that $y(0) = 0 \Rightarrow c_0 = 0$ while $y'(0) = 1 \Rightarrow c_1 = 1$. Since $c_0 = 0$, it follows that $c_2 = 0$, $c_4 = 0$, and in general $c_{2k} = 0$ where $k$ is a non-negative integer.
Since $c_1 = 1$, $c_3 = \frac{2-\lambda}{6} \cdot 1 = \frac{2-\lambda}{6}$. In turn, $c_5 = \frac{1}{20} \cdot (6-\lambda) \cdot c_3 = \frac{(6-\lambda)(2-\lambda)}{120}$.
We conclude that $y_1 = x + \frac{2-\lambda}{6}x^3 + \frac{(6-\lambda)(2-\lambda)}{120}x^5 + \cdots = x + \sum_{n=1}^{\infty} \frac{(2-\lambda)(6-\lambda)\cdots(4n-2-\lambda)}{(2n+1)!}x^{2n+1}$. (It may take some additional exploration to convince you of the general term, but don't worry too much about it; we're only asked for three non-zero terms.)

b. For $y_2$ we have the same relations between the coefficients, just different initial conditions. For this function, $y(0) = 1$, which implies that $c_0 = 1$. Further, $y'(0) = 0$ which implies that $c_1 = 0$ and, in turn, $c_{2k+1} = 0$ for any non-negative integer $k$. $c_2 = \frac{-\lambda c_0}{2} = \frac{-\lambda}{2}$. It follows that $c_4 = \frac{1}{12} \cdot (4-\lambda)c_2 = \frac{-\lambda(4-\lambda)}{24}$ and $c_6 = \frac{1}{30} \cdot (8-\lambda) \cdot c_4 = \frac{-\lambda(4-\lambda)(8-\lambda)}{6!}$.
Therefore $y_2 = 1 - \frac{\lambda}{2!}x^2 - \frac{\lambda(4-\lambda)}{4!}x^4 + \cdots = x - \sum_{n=1}^{\infty} \frac{\lambda(4-\lambda)\cdots(4n-4-\lambda)}{(2n)!}x^{2n}$. (Again, the general term is just for extra fun.)

c. If $\lambda = 4$, it will be $y_2$ that terminates; every term in the series with a factor of $(4 - \lambda)$ will vanish. Therefore we will have simply $y_2 = 1 - \frac{4}{2!}x^2$ or $h_4(x) = 1 - 2x^2$.
If $\lambda = 6$, then it will be $y_1$ that terminates. In this case, we have $y_1 = x + \frac{2-6}{6}x^3$ or $h_6(x) = x - \frac{2}{3}x^3$.

d. $H_2(x) = k \cdot h_4(x) = k \cdot \left(1 - 2x^2\right)$. We choose $k$ so that the leading coefficient will be $2^2 = 4$. Since the leading coefficient of $h_4(x)$ is -2, we scale by -2.
$H_2(x) = -2\left(1 - 2x^2\right) = 4x^2 - 2$.
$H_3(x) = k \cdot h_6(x) = k \cdot \left(x - \frac{2}{3}x^3\right)$. We choose $k$ so that the leading coefficient will be $2^3 = 8$, requiring us to scale up by $\frac{8}{-2/3} = -12$.
$H_3(x) = -12\left(x - \frac{2}{3}x^3\right) = 8x^3 - 12x$.

e. $H_2(x) = 4x^2 - 2 \Rightarrow H_2'(x) = 8x \Rightarrow H_2''(x) = 8$. In this case, the left side of the Hermite equation takes on the form $8 - 2x \cdot 8x + 4\left(4x^2 - 2\right)$. (Remember that $\lambda = 4$.) Simplifying…

$$\begin{aligned}8 - 2x \cdot 8x + 4\left(4x^2 - 2\right) &= 8 - 16x^2 + 16x^2 - 8 \\ &= 0\end{aligned}$$

It checks, which is hopefully not a surprise.
$H_3(x) = 8x^3 - 12x \Rightarrow H_3'(x) = 24x^2 - 12 \Rightarrow H_3''(x) = 48x$. Noting that $\lambda = 6$ for $H_3(x)$, the left side of the Hermite equation takes on the form $48x - 2x \cdot \left(24x^2 - 12\right) + 6 \cdot \left(8x^3 - 12x\right)$.
Simplifying…

$$48x-2x\cdot\left(24x^2-12\right)+6\cdot\left(8x^3-12x\right)=48x-48x^3+24x+48x^3-72x$$
$$=72x-72x-48x^3+48x^3$$
$$=0$$

This solution also checks out.

56. We know that $e^x=1+x+\frac{1}{2!}+\cdots+\frac{x^n}{n!}+\cdots$. Specifically, $e=e^1=1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\cdots$. All the terms in this series for $e$ are positive. Therefore any partial sum of the series will be an underestimate of the value of $e$. Taking the first two terms and grouping the rest as the "tail," we have $e$ = 2 + tail. Since the tail must be positive, we have $e > 2$.
To get an upper bound for $e$, we will compare the series to one that is greater and whose sum we can determine. A good candidate is $\sum\frac{1}{2^n}$. For all $n\geq 4$, $n!>2^n$. Therefore, for all such $n$, $\frac{1}{n!}<\frac{1}{2^n}=\left(\frac{1}{2}\right)^n$.
Since $\frac{1}{n!}<\left(\frac{1}{2}\right)^n$ for all $n\geq 4$, it follows that $\sum_{n=4}^{\infty}\frac{1}{n!}<\sum_{n=4}^{\infty}\left(\frac{1}{2}\right)^n$. (To be totally rigorous, we would need to justify this statement a bit more, but it is hopefully plausible enough for our purposes.) However, $e$ is not only $\sum_{n=4}^{\infty}\frac{1}{n!}$; there are some terms missing. $e=1+1+\frac{1}{2!}+\frac{1}{3!}+\sum_{n=4}^{\infty}\frac{1}{n!}$. Adding the missing terms to both sides of the inequality above we have $1+1+\frac{1}{2!}+\frac{1}{3!}+\sum_{n=4}^{\infty}\frac{1}{n!}<1+1+\frac{1}{2!}+\frac{1}{3!}+\sum_{n=4}^{\infty}\left(\frac{1}{2}\right)^n$ or
$e=\frac{8}{3}+\sum_{n=4}^{\infty}\frac{1}{n!}<\frac{8}{3}+\sum_{n=4}^{\infty}\left(\frac{1}{2}\right)^n$. Of course, we can actually evaluate the series on the right side of the inequality since it is geometric. $\sum_{n=4}^{\infty}\left(\frac{1}{2}\right)^n=\frac{(1/2)^4}{1-1/2}=\frac{1}{8}$. Therefore $e<\frac{8}{3}+\frac{1}{8}=\frac{8}{3}=\frac{67}{24}$, or about 2.7916666….
This is certainly less than the 4 we were asked to show. We have done better. We have shown that $2<e<3$.

57. a. This series is geometric with common ratio 1/2 and initial term 3. The sum is $\frac{3}{1-\frac{1}{2}}=6$.
    b. This series is the arctangent series (given away by the alternation and the odd, non-factorial denominators) with 1 plugged in for $x$. Its sum is arctan(1) or $\frac{\pi}{4}$.
    c. This is series is based on the series for $e^x$ (given away by the factorial denominators). The powers of 3 suggest that 3 has been plugged in for $x$. Indeed, the series sums to $e^3$.
    d. If we rewrite the series as $-\frac{(1/2)}{1}-\frac{(1/4)}{2}-\frac{(1/8)}{3}-\frac{(1/16)}{4}-\cdots=-\frac{(1/2)^1}{1}-\frac{(1/2)^2}{2}-\frac{(1/2)^3}{3}-\frac{(1/2)^4}{4}-\cdots$, things become clearer. Now we see that we have increasing powers of 1/2 and that the $n^{\text{th}}$ denominator is just $n$. This is the series for $\ln(1-x)$ with 1/2 plugged in for $x$. Therefore its sum is $\ln\left(1-\frac{1}{2}\right)=\ln\frac{1}{2}$.

58. a. The alternation and odd, factorial denominators suggest the sine series, the first term of which is $x$. The first term of the given series is -5, which means that -5 has been plugged in for $x$. Subsequent powers of -5 in the other terms confirm this. This series sums to sin(-5).
    b. The denominators are factorials, so this is based on the series for $e^x$. Since the terms alternate, we might suspect that -1 has been plugged in for $x$. But that's not quite right. $e^{-1}=1-1+\frac{1}{2}-\frac{1}{6}+\cdots$, or $e^{-1}=\frac{1}{2}+\frac{1}{6}+\cdots$ after canceling the first two ones. The given series has an extra 1, so its value is simple $1+\frac{1}{e}$.
    c. This series is geometric with initial term 216 and common ratio -1/6. Its sum is $\frac{216}{1-(-1/6)}=\frac{1296}{7}$ or $185\frac{1}{7}$.

d. This is the exponential series with ln(4) plugged in for *x*. Its sum is $e^{\ln 4} = 4$.

59. a. We apply the ratio test to $\sum \frac{n}{|r|^n}$ which is a positive-term series (and the absolute-value version of the series $\sum \frac{n}{r^n}$). $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{n+1}{|r|^{n+1}} \cdot \frac{|r|^n}{n} = \lim\limits_{n\to\infty} \frac{n+1}{n} \cdot \frac{|r|^n}{|r|^{n+1}} = \frac{1}{|r|}$. If $|r| > 1$, then $\frac{1}{|r|} < 1$. Therefore $\sum \frac{n}{|r|^n}$ converges by the ratio test when $|r| > 1$. This means that $\sum \frac{n}{r^n}$ converges absolutely when $|r| > 1$. If $|r| < 1$, then $\frac{1}{|r|} > 1$, and we have divergence by the ratio test. If $|r| = 1$, the series simplifies to $\sum n$ which clearly diverges. We conclude that this series converges iff $|r| > 1$.

b. $f(x) = \frac{1}{1-x} \Rightarrow f'(x) = \frac{1}{(1-x)^2}$. Then $g(x) = x \cdot f'(x) = \frac{x}{(1-x)^2}$.

c. The Maclaurin series for $f(x)$ is $1 + x + x^2 + \cdots + x^n + \cdots$. Term-by-term differentiation gives $f'(x) = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots$. Finally, multiplying through by *x* gives $g(x) = x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots$.

d. To find the interval of convergence for $g(x)$, we apply the ratio test to the absolute value of the general term. $\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim\limits_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{nx^n} \right| = \lim\limits_{n\to\infty} \left| \frac{n+1}{n} \cdot x \right| = |x|$. Therefore this series converges if $|x| < 1$. We ignore the endpoints since we are not asked to deal with them. (The series diverges at both endpoints.)

e. We know that the series for $g(x)$ converges when $|x| < 1$. If *r* is greater than 1 in magnitude, then $\frac{1}{r}$ certainly falls within this interval of convergence. Therefore it is meaningful to say that

$$g\left(\tfrac{1}{r}\right) = \sum_{n=1}^{\infty} n\left(\tfrac{1}{r}\right)^n = \sum_{n=1}^{\infty} \frac{n}{r^n}.$$

However, we also have an explicit expression for $g(x)$. $g(x) = \frac{x}{(1-x)^2}$. Therefore $g\left(\frac{1}{r}\right) = \frac{1/r}{(1-1/r)^2}$.

Cleaning this up, we have $g\left(\frac{1}{r}\right) = \frac{1/r}{\left(1-\frac{1}{r}\right)^2} \cdot \frac{r^2}{r^2} = \frac{r}{\left[\left(1-\frac{1}{r}\right)\cdot r\right]^2} = \frac{r}{(r-1)^2}$ as desired.

60. a. $f(x) = \frac{e^x - 1}{x} \Rightarrow f'(x) = \frac{xe^x - \left(e^x - 1\right)}{x^2} = \frac{xe^x - e^x + 1}{x^2}$. $f'(1) = \frac{1e - e + 1}{1^2} = 1$.

b. $f(x) = \dfrac{\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right) - 1}{x} = \dfrac{x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots}{x} = 1 + \frac{1}{2!}x + \frac{1}{3!}x^2 + \cdots + \frac{1}{(n+1)!}x^n + \cdots$.

Differentiating gives $f'(x) = \frac{1}{2!} + \frac{2}{3!}x + \cdots + \frac{n}{(n+1)!}x^{n-1} + \cdots$.

c. From the series in part (b), we have $f'(1) = \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} + \cdots = \sum\limits_{n=1}^{\infty} \frac{n}{(n+1)!}$. We also found in part (a) that $f'(1) = 1$. Therefore $\sum\limits_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$.

61. a. $f(1) = \int_0^1 te^t\,dt = \left(te^t - e^t\right)\Big|_0^1 = 1e^1 - e^1 - \left(0e^0 - e^0\right) = e - e + 1 = 1$

b. $e^t = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots$

$$te^t = t + t^2 + \tfrac{1}{2!}t^3 + \tfrac{1}{3!}t^4 + \cdots$$

$$\int_0^x te^t\,dt = \int_0^x \left(t + t^2 + \tfrac{1}{2!}t^3 + \tfrac{1}{3!}t^4 + \cdots\right)dt = \left(\tfrac{1}{2\cdot 0!}t^2 + \tfrac{1}{3\cdot 1!}t^3 + \tfrac{1}{4\cdot 2!}t^4 + \tfrac{1}{5\cdot 3!}t^5 + \cdots\right)\Big|_0^x$$

$$= \tfrac{1}{2\cdot 0!}x^2 + \tfrac{1}{3\cdot 1!}x^3 + \tfrac{1}{4\cdot 2!}x^4 + \tfrac{1}{5\cdot 3!}x^5 + \cdots$$

$$f(x) = \sum_{n=0}^{\infty} \tfrac{1}{(n+2)\cdot n!}x^{n+2}$$

c. From part (b), we have $f(1) = \sum_{n=0}^{\infty} \tfrac{1}{(n+2)\cdot n!}\cdot(1)^{n+2} = \sum_{n=0}^{\infty} \tfrac{1}{(n+2)\cdot n!}$. But from part (a), we have $f(1) = 1$.

Therefore $\sum_{n=0}^{\infty} \tfrac{1}{(n+2)\cdot n!} = 1$.

62. A little re-indexing and algebra shows that the series are the same:

$$\sum_{n=1}^{\infty} \tfrac{n}{(n+1)!} = \sum_{n=0}^{\infty} \tfrac{n+1}{(n+2)!} = \sum_{n=0}^{\infty} \tfrac{n+1}{(n+2)\cdot(n+1)\cdot n!} = \sum_{n=0}^{\infty} \tfrac{1}{(n+2)\cdot n!}.$$

63. a. $1 + 2x + 3x^2 + x^3 + 2x^4 + 3x^5 + \cdots = \left(1 + x^3 + x^6 + \cdots\right) + 2\left(x + x^4 + x^7 + \cdots\right) + 3\left(x^2 + x^5 + x^8 + \cdots\right)$

(As long as $|x| < 1$, these three geometric series are absolutely convergent. Therefore, we can reorder the summation without fear of changing its sum.)
The three geometric series each have common ratio $x^3$; they differ only in their initial term.

$1 + x^3 + x^6 + \cdots = \tfrac{1}{1-x^3}$

$2\left(x + x^4 + x^7 + \cdots\right) = 2\cdot\tfrac{x}{1-x^3} = \tfrac{2x}{1-x^3}$

$3\left(x^2 + x^5 + x^8 + \cdots\right) = 3\cdot\tfrac{x^2}{1-x^3} = \tfrac{3x^2}{1-x^3}$.

The sum of the series is therefore $f(x) = \tfrac{1}{1-x^3} + \tfrac{2x}{1-x^3} + \tfrac{3x^2}{1-x^3} = \tfrac{1+2x+3x^2}{1-x^3}$.

b. Each of the individual geometric series has an interval of convergence of $-1 < x < 1$. By the hint, it follows that the sum has this same interval of convergence. The radius of convergence of this series is 1.

c. $1 + \tfrac{2}{2} + \tfrac{3}{4} + \tfrac{1}{8} + \tfrac{2}{16} + \tfrac{3}{32} + \cdots = f\left(\tfrac{1}{2}\right) = \tfrac{1+2(1/2)+3(1/2)^2}{1-(1/2)^3} = \tfrac{22}{7}$

$1 - \tfrac{2}{3} + \tfrac{3}{9} - \tfrac{1}{27} + \tfrac{2}{81} - \tfrac{3}{243} + \cdots = f\left(\tfrac{-1}{3}\right) = \tfrac{1+2(-1/3)+3(-1/3)^2}{1-(-1/3)^3} = \tfrac{9}{14}$

d. This sum is much like the original sum, except there will be a total of *m* interwoven series, each with a common ratio of $x^m$. In analogy to part (a), the sum will be $\tfrac{1}{1-x^m} + \tfrac{2x}{1-x^m} + \tfrac{3x^2}{1-x^m} + \cdots + \tfrac{mx^{m-1}}{1-x^m}$, or $\tfrac{1+2x+3x^2+\cdots+mx^{m-1}}{1-x^m}$. Since each geometric series still converges on $-1 < x < 1$, that holds for the "whole" series as well. The radius of convergence of the whole series is 1.

e. The only change that is made here is in the coefficients, but these coefficients will still repeat cyclically. What used to be a 1 is now $c_0$, what used to be a 2 is now $c_1$, etc. The sum of the series is now $\tfrac{c_0+c_1x+c_2x^2+\cdots+c_{m-1}x^{m-1}}{1-x^m}$.

f. The coefficients here are cycling with a period of 4, and they cycle through the numbers 2, 3, 5, and 8. In series form our function is $f(x) = 2 + 3x + 5x^2 + 8x^3 + 2x^4 + 3x^5 + 5x^6 + 8x^7 + \cdots$. Explicitly, the function is $f(x) = \tfrac{2+3x+5x^2+8x^3}{1-x^4}$. The powers of 5 in the denominator arise from plugging in 1/5 to the series. We can therefore evaluate this sum as

$$f\left(\tfrac{1}{5}\right)=\frac{2+3\left(\tfrac{1}{5}\right)+5\left(\tfrac{1}{5}\right)^2+8\left(\tfrac{1}{5}\right)^3}{1-\left(\tfrac{1}{5}\right)^4}=\frac{895}{312}=2.8685897\ldots.$$

64. $f(x)=\frac{1}{x}=\frac{1}{1+(x-1)}=\frac{1}{1-^-(x-1)}$ is a geometric series with common ratio $-(x-1)$ and initial term 1.

Therefore $f(x)=\sum_{n=0}^{\infty}\left(-(x-1)\right)^n=\sum_{n=0}^{\infty}(-1)^n(x-1)^n$. The series converges iff $\left|(-1)\cdot(x-1)\right|<1$.

$\left|(-1)\cdot(x-1)\right|<1\Rightarrow -1<x-1<1\Rightarrow 0<x<2$

The interval of convergence of this series is $0 < x < 2$.

65. $f(x)=\frac{1}{x}=\frac{1}{4+(x-4)}=\frac{1}{4}\cdot\frac{1}{1+\frac{x-4}{4}}=\frac{1}{4}\cdot\frac{1}{1-\frac{-(x-4)}{4}}$. This is a geometric series with initial term 1/4 and common ratio $\frac{-(x-4)}{4}$. Therefore $f(x)=\sum_{n=0}^{\infty}\frac{1}{4}\cdot\left(\frac{-(x-4)}{4}\right)^n$.

Graph some partial sums of the series from Problems 64 and 65 along with $f(x)=\frac{1}{x}$ to see what's going on here.

66. a. $f(x)=\frac{1}{1-x}=\frac{1}{-1-(x-2)}=-1\cdot\frac{1}{1+(x-2)}=\frac{-1}{1-^-(x-2)}$ is geometric with initial term -1 and common ratio $-(x-2)$. It can be expanded as a series as $\sum_{n=0}^{\infty}-1\cdot\left(-(x-2)\right)^n=\sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n$.

b. $f(x)=\frac{1}{1-x}=\frac{1}{3-(x+2)}=\frac{1}{3}\cdot\frac{1}{1-\frac{x+2}{3}}$ is geometric with initial term 1/3 and common ratio $\frac{x+2}{3}$. It can be expanded as $\sum_{n=0}^{\infty}\frac{1}{3}\cdot\left(\frac{x+2}{3}\right)^n$.

c. $\frac{1}{1-x}=\frac{1}{-4-(x-5)}=\frac{-1}{4}\cdot\frac{1}{1+\frac{x-5}{4}}$ is geometric with initial term -1/4 and common ratio $\frac{-(x-5)}{4}$. It can be expanded as $\sum_{n=0}^{\infty}\frac{-1}{4}\cdot\left(\frac{-(x-5)}{4}\right)^n=\frac{1}{4}\cdot\sum_{n=0}^{\infty}(-1)^{n+1}\cdot\left(\frac{x-5}{4}\right)^n$.

d. $\frac{1}{1-x}=\frac{1}{(1-k)-(x-k)}=\frac{1}{1-k}\cdot\frac{1}{1-\frac{x-k}{1-k}}$ is geometric with initial term $\frac{1}{1-k}$ and common ratio $\frac{x-k}{1-k}$. It can be expanded as $\sum_{n=0}^{\infty}\frac{1}{1-k}\cdot\left(\frac{x-k}{1-k}\right)^n$.

67. $f(x)=\frac{1}{3-x}=\frac{1}{1-(x-2)}$ is geometric with initial term 1 and common ratio $(x-2)$. It can be expanded as $\sum_{n=0}^{\infty}(x-2)^n$.

68. a. It is clear that $f(x)$ is differentiable for all $x$ other than $\pm\pi$; the functions cos($x$) and -1 are infinitely differentiable, so we only have to worry about the $x$-values where the two functions "meet up."

$f'(\pi)=\lim_{x\to\pi}\frac{f(x)-f(\pi)}{x-\pi}$, provided this limit exists. As $x\to\pi^+$, $f(x)=-1$. So we have $\lim_{x\to\pi^+}\frac{-1-^-1}{x-\pi}=\lim_{x\to\pi^+}\frac{0}{x-\pi}=0$. As $x\to\pi^-$, $f(x)=\cos(x)$. The limit is now $\lim_{x\to\pi}\frac{\cos(x)-^-1}{x-\pi}=\lim_{x\to\pi}\frac{\cos x+1}{x-\pi}=\lim_{x\to\pi}\frac{\sin x}{1}=0$. Since the two one-sided limits agree, $f'(\pi)$ is defined and is zero.

The computations work out to be identical for $f'(-\pi)$. It is also defined and equal to zero.

We conclude that $f$ is differentiable for all $x$.

b. Since we are looking at $x = 0$, we use $\cos(x)$ as the rule for $f(x)$. If we do this, we will simply generate our old friend $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ as the Maclaurin series for $f(x)$.

c. We have already seen (Problem 1) that the series in part (b) converges. Furthermore, we showed in Problem 2 that this series converges *to the cosine function* for all $x$. That's great on the interval $-\pi < x < \pi$. However, for $|x| > \pi$, $f(x)$ is not the cosine function. Therefore this series cannot converge to $f(x)$ for all $x$.

d. While we showed in part (a) that $f$ is differentiable for all $x$, $f$ is only once-differentiable for all $x$. All higher-order derivatives of $f$ are undefined at $x = \pm\pi$. We should only expect the Macluarin series to converge to $f(x)$ in the interval $-\pi < x < \pi$, which in fact it does.

69\. a. Before diving into a detailed argument, it might be helpful to compute a few derivatives to see—at least roughly—what is going on. Our function is

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Basic differentiation rules show that $f'(x) = \frac{2}{x^3} \cdot e^{-1/x^2}$ for $x \neq 0$. To determine $f'(0)$, we use the definition of the derivative.

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{e^{-1/x^2}}{x} = \lim_{x\to 0} \frac{1/x}{e^{1/x^2}}$$

We can use l'Hospital's rule here, but it will make sense for what's coming down the road to do something different. We introduce a new variable $u$ so that $u = \frac{1}{x}$. As $x \to 0$, $u \to \pm\infty$. (The ± depends on whether $x$ approaches 0 from the left or the right. Ultimately this will only effect whether our limits approach 0 from above or below, so it's not that big a deal.) Under this change of variables, our limit becomes $\lim_{u\to\pm\infty} \frac{u}{e^{u^2}}$. $\lim_{u\to\pm\infty} \frac{u}{e^{u^2}} = 0$. As claimed above, it should be clear that this limit is zero regardless of whether $u \to \infty$ or $u \to -\infty$; the exponential growth will dominate the polynomial growth (in this case, linear growth) in both directions. In any event, we have just proved that $f'(0) = 0$. This means that

$$f'(x) = \begin{cases} e^{-1/x^2} \cdot \dfrac{2}{x^3}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

What about $f''(x)$? For $x \neq 0$, ordinary differentiation rules show that $f''(x) = e^{-1/x^2}\left(\frac{4}{x^6} - \frac{6}{x^4}\right)$. For $x = 0$, we again need a limit.

$$f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0} \frac{e^{-1/x^2} \cdot \frac{2}{x^3} - 0}{x} = \lim_{x\to 0} \frac{2e^{-1/x^2}}{x^4}$$

We use our same trick as in the last limit.

$$f''(0) = \lim_{x\to 0} \frac{2e^{-1/x^2}}{x^4} = \lim_{x\to 0} \frac{2/x^4}{e^{1/x^2}} = \lim_{u\to\pm\infty} \frac{2u^4}{e^{u^2}} = 0$$

We now have

$$f''(x) = \begin{cases} e^{-1/x^2}\left(\frac{4}{x^6} - \frac{6}{x^4}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

I'll give you one more derivative for free.

$$f'''(x)=\begin{cases} e^{-1/x^2}\cdot\left(\frac{8}{x^9}-\frac{36}{x^7}+\frac{24}{x^5}\right), & x\neq 0\\ 0, & x=0\end{cases}$$

At this point, you may be suitably convinced that $f^{(k)}(0)=0$ for all non-negative integers $k$. One can continue to take derivatives like this *ad nauseum*. While the forms of the derivatives will change (getting uglier and uglier), the basic process is the same; only the details differ.
Or we can give a more rigorous proof. I have seen a few proofs for the fact that $f^{(k)}(0)=0$, but none is especially easy. The argument that follows is based on the approach in Michael Spivak's text *Calculus*. Before beginning, let's see if we can take some lessons from what we have done so far. First, based on the limit computations we have done, it seems that it will be useful to claim that $\lim_{x\to 0}\frac{e^{-1/x^2}}{x^m}=0$ for all non-negative integers $m$. Second, it appears that the $k^{\text{th}}$ derivative of $f$, for $x\neq 0$, is given by $f^{(k)}(x)=e^{-1/x^2}\cdot(\text{a sum of inverse power functions})$. If you are particularly observant, you might have observed that the highest-degree of the inverse power functions appeared to be $3k$. In other words, it seems that for $x\neq 0$

$$f^{(k)}(x)=e^{-1/x^2}\cdot\sum_{i=1}^{3k}\frac{a_i}{x^i}$$

for some set of numbers $a_i$.

Our strategy will be to prove the first thing—the statement about the limit—as a lemma. Then we will prove the general form of $f^{(k)}(x)$ for non-zero $x$ by induction. Finally, we will prove that $f^{(k)}(0)=0$ by induction; the proof will rely on both of the previous steps. Here we go.

Lemma: $\lim_{x\to 0}\dfrac{e^{-1/x^2}}{x^k}=0$ for any non-negative integer $k$.

Proof of lemma: $\lim_{x\to 0}\dfrac{e^{-1/x^2}}{x^k}=\lim_{x\to 0}\dfrac{1/x^k}{e^{1/x^2}}=\lim_{u\to\pm\infty}\dfrac{u^k}{e^{u^2}}$, where $u=1/x$. Since the exponential denominator will dominate the polynomial numerator, no matter how large $k$ is, this limit is 0 as $u$ goes both to $+\infty$ and $-\infty$. We conclude that $\lim_{x\to 0}\dfrac{e^{-1/x^2}}{x^k}=0$.

Next we will prove by induction that when $x$ is not zero, $f^{(k)}(x)=e^{-1/x^2}\cdot\sum_{i=1}^{3k}\frac{a_i}{x^i}$ for some set of numbers $a_i$. For the entirety of this inductive argument, it will be assumed that $x\neq 0$.
The base case will be for $k$ = 1. As we saw above, $f'(x)=e^{-1/x^2}\cdot\frac{2}{x^3}$. This is of the desired form. Notice that $a_1=a_2=0$ and $a_3=2$. This verifies the base case.
For the inductive step, we will assume that $f^{(k)}(x)=e^{-1/x^2}\cdot\sum_{i=1}^{3k}\frac{a_i}{x^i}$ for some set of numbers $a_i$. We will show that $f^{(k+1)}(x)=e^{-1/x^2}\cdot\sum_{i=1}^{3k+3}\frac{b_i}{x^i}$ for some *other* set of numbers $b_i$.
If $f^{(k)}(x)=e^{-1/x^2}\cdot\sum_{i=1}^{3k}\frac{a_i}{x^i}$, then differentiation gives the following.

$$\begin{aligned} f^{(k+1)}(x) &= \frac{2}{x^3} e^{-1/x^2} \cdot \sum_{i=1}^{3k} \frac{a_i}{x^i} + e^{-1/x^2} \cdot \sum_{i=1}^{3k} \frac{-ia_i}{x^{i+1}} \\ &= e^{-1/x^2} \cdot \left[ \frac{2}{x^3} \cdot \sum_{i=1}^{3k} \frac{a_i}{x^i} + \sum_{i=1}^{3k} \frac{-ia_i}{x^{i+1}} \right] \\ &= e^{-1/x^2} \cdot \left[ \sum_{i=1}^{3k} \frac{2a_i}{x^{i+3}} + \sum_{i=1}^{3k} \frac{-ia_i}{x^{i+1}} \right] \\ &= e^{-1/x^2} \cdot \sum_{i=1}^{3k} \left( \frac{2a_i}{x^{i+3}} + \frac{-ia_i}{x^{i+1}} \right) \end{aligned}$$

Notice that since the summation runs to $i = 3k$, the denominators in first fraction run to $3k + 3$. By taking this into account and regrouping the fractions with some unspecified numerators $b_i$, we arrive at $f^{(k+1)}(x) = e^{-1/x^2} \cdot \sum_{i=1}^{3k+3} \left( \frac{b_i}{x^i} \right)$ for $x \neq 0$ as desired. This completes the inductive proof.

We are finally in a position to prove what we are really after: $f^{(k)}(0) = 0$ for all non-negative integers $k$. We will again proceed by induction on $k$. The base case is $k = 0$. By definition, $f^{(0)}(0) = f(0) = 0$. If you find the base case trivial, we have also already verified that $f^{(1)}(0) = f'(0) = 0$.

For the inductive step, we will assume that $f^{(k)}(0) = 0$ and show that $f^{(k+1)}(0) = 0$.

By definition, $f^{(k+1)}(0) = \lim_{x \to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0}$. The second term in the numerator is zero by the inductive hypothesis. The first term is, as we have just proved, $f^{(k)}(x) = e^{-1/x^2} \cdot \sum_{i=1}^{3k} \frac{a_i}{x^i}$ for some set of numbers $a_i$.

$$\begin{aligned} f^{(k+1)}(0) &= \lim_{x \to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{e^{-1/x^2} \cdot \sum_{i=1}^{3k} \frac{a_i}{x^i}}{x} \\ &= \lim_{x \to 0} \left( e^{-1/x^2} \cdot \sum_{i=1}^{3k} \frac{a_i}{x^{i+1}} \right) \\ &= \sum_{i=1}^{3k} \left( \lim_{x \to 0} \frac{a_i e^{-1/x^2}}{x^{i+1}} \right) = \sum_{i=1}^{3k} 0 \\ &= 0 \end{aligned}$$

The immense simplification to the summation in the penultimate line is due to the lemma. This completes the proof!

b. Since $g^{(n)}(0) = 0$ for all $n$, the Maclaurin series for $g$ is simply the zero function: $M(x) = 0$.

c. $g(x)$ is itself equal to 0 only at $x = 0$. For all other $x$, $g(x)$ is positive. Therefore the Maclaurin series for $g$ does not equal the values of the function anywhere other than $x = 0$. There is no interval on which the series converges to $g(x)$.

70. a. The Maclaurin series is centered at $x = 0$, and the break in the domain of $h$ is at $x = 2$, two units away. We would initially expect the radius of convergence for the Maclaurin series to be 2.

b. At $x = 0$, $h(x) = x + 2$ (factor and cancel). Therefore the Maclaurin series for $h(x)$ is simply $2 + x$. This trivially converges for all $x$ (even $x = 2$).
This example shows that a removable discontinuity is not what is sometimes called an "essential singularity" of a function. Essential singularities always limit the radius of convergence of a power series for a function while removable discontinuities do not.

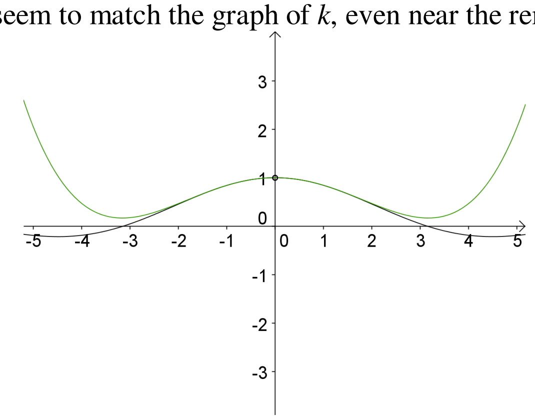
71. a. While $k(0)$ is undefined, $\lim_{k\to 0} \frac{\sin x}{x} = 1$. Therefore the discontinuity at $x = 0$ is removable.

b. $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5}x^5 - \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots$

$\frac{\sin x}{x} = 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n}$ .

c. $k(x)$ is not defined at $x = 0$ and is therefore certainly not differentiable at $x = 0$. By definition, a Maclaurin series is for a function that is infinitely differentiable at $x = 0$.

d. Below are the graphs of $k$ and (in green) the fourth-degree partial sum from part (b). As you see, the polynomial does seem to match the graph of $k$, even near the removable discontinuity at $x = 0$.




$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}x^{2n+2}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{(2n+1)!}{(2n+3)!} \cdot \frac{x^{2n+2}}{x^{2n}} \right| = \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)} = 0 < 1$ for all $x$. This power series converges for all $x$, and indeed matches $k(x)$ perfectly everywhere except at $x = 0$… the center of the series.